INTRODUCTION

TO THE

ELEMENTARY FUNCTIONS

BY

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WITH THE EDITORIAL COÖPERATION OF

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PREFACE

This book is an attempt to solve the problem of the first-year collegiate course in mathematics. That the problem is a very real one is attested by the many discussions constantly taking place among teachers of mathematics and others interested in education. The traditional Freshman course, consisting of "college algebra," trigonometry, and solid geometry or elementary analytic geometry, is very generally regarded as unsatisfactory.

There are three main objections to this traditional course: first, it is not unified, so that it sacrifices time and fails to hold the student's interest; secondly, much of the subject matter should come after a first course in calculus, when it would gain vastly in significance; thirdly, the usual plan has deprived the large majority of college students of any introduction to the calculus, which is the heart and soul of modern mathematics and natural science. Only that small number electing to go beyond the first year of collegiate mathematics have the opportunity to become acquainted with the subject, which unquestionably represents one of the most important lines of development of human thought during the past two centuries.

Accordingly, we decided to construct a course with the fundamental idea of functionality as its unifying principle, and leading up to some elementary work in calculus as its culmination. The advantages of this arrangement are that it not only meets the objections stated in the preceding paragraph, but saves time by avoiding the repetition inevitable in the triple arrangement of subjects; and, what is more important, it leads to a deeper understanding of the significance of mathematical principles and relations than the student is likely to gain through the traditional course. Thus, whether he goes farther into the study of

mathematics or has but one year for the study, we believe that he will be the gainer by taking the unified course.

The course presented in this book is the result of our experience in Grinnell College, mimeographed copies having been used and revised during five successive years. The material included comprises the simpler and more important parts of plane trigonometry and analytic geometry, followed by an introduction to the differential calculus, including differentiation of the simpler algebraic functions and applications to problems of rates and maxima and minima. The conic sections are not studied as extensively as in most textbooks of analytic geometry, but enough has been given to make the student feel familiar with these important curves.

The trigonometric functions are introduced early, and the general definitions are given at once, instead of those valid only for the acute angle. Thus the connection with the coördinate system is established from the first, and a clearer idea of the meaning of the trigonometric functions is obtained than if the student's attention is for some time confined to the case of the acute angle. The applications of the trigonometric functions to the solution of right triangles and problems depending upon them is made without the use of logarithms, as experience shows that the early introduction of logarithms may easily lead to mechanical methods of work.

The number of numerical exercises is not large, as the teacher can easily supplement those in the text by as many others as he wishes. We feel that the purely computational work can easily be overdone in the first-year course. Four-place tables may be used in this work; the Wentworth-Smith tables, or others of like nature, are very satisfactory.

The arrangement is almost exclusively inductive, and the style direct and informal throughout. The explanations are not always as detailed as in many texts, the object being to lead the student to supply the connecting links for himself, where they are not explicitly given in the text. Moreover, many of the important results are altogether left to the student as exercises. In such cases, as in the case of all the most important formulas

throughout the book, attention is called to their importance by the use of black type.

The course will be found suitable for an advanced course in the secondary school, as well as for the first year in college, and in this case it might very well be made a five-hour course. For the student who has somewhat lost touch with his previous work in mathematics, a small amount of review matter has been placed in appendixes. If this work has to be taken, it will probably be unwise to attempt to cover all of the text, and certain paragraphs have accordingly been starred, to indicate that they may be omitted without interfering with the unity of the course.

We have not gone so far in the way of radical changes in subject matter as our personal feelings would lead us, because we believe that progress in the teaching of mathematics, as in everything else, should be in the nature of evolution rather than revolution. For instance, no work in the integral calculus is given, although we firmly believe that this topic should eventually be included in the first-year course; but it seemed to require a greater departure from the traditional course than is as yet practicable. We hope that the present book may prove a contribution to the solution of the problem presented by the first year of college mathematics, and that experience will indicate what further steps may advantageously be taken.

R. B. M.
W. J. R.

CONTENTS

THE ELEMENTARY FUNCTIONS

CHAPTER I

THE GRAPHICAL REPRESENTATION

1. Measurement. The study of algebra is concerned with *numbers*; that of geometry, with *space*. The object of this chapter is to bring into closer relation these two portions of elementary mathematics, in order that each may be used to help the other. The simplest process in which number and space are combined is *measurement*.

2. The important rôle which measuring plays in the work of the world is so apparent to everyone that it needs no mention. The nature of this important process, as applied to distances, is made clear by the following considerations: If we wish to measure a line segment (that is, a limited portion of a straight line) AB, we begin by choosing any convenient line segment CD, which we call a *unit*; we then apply CD to AB so that C coincides with A, when D will fall upon E, a point between A and B, in case AB is $> CD$. Thus $AE = CD$. We next apply CD to EB, C coinciding now with E, and D falling at F. Thus $EF = CD$, and $AF = 2 \cdot CD$. By continuing this process again and again we may eventually find that D falls upon B, let us say, the nth time we applied CD successively. Then $AB = n \cdot CD$, and we say *the length of AB is n units.* We also say the unit is *contained exactly* in the segment AB. Whenever this happens, the length of the segment is an *integer*.

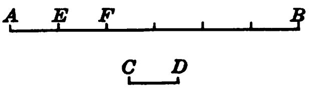


FIG. 1

But in practice this does not usually happen. More often the unit will be contained exactly, let us say n times, in a segment AB', less than AB, whereas the remainder $B'B$ of AB is less than

the unit CD (Fig. 2). We can then assert that the length of AB is greater than n units but less than $n+1$. That is, we have measured the segment *AB to the nearest integer* less than its length. If we wish to get a closer approximation, we may divide the unit CD into any number of parts, as k, and proceed to measure the segment $B'B$, using one of these *parts* of CD as a new unit. If this part of the unit is exactly contained in $B'B$, say m times, then $B'B = m$ $k$ths of a unit, and the length of AB equals $n + \frac{m}{k}\cdot$ This number can be written $\frac{nk+m}{k}$, that is, in the form of the *quotient* of two *integers.*

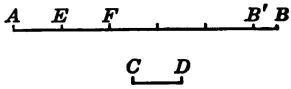


FIG. 2

If it happens that the kth part of CD is not contained exactly in $B'B$, m of these parts being less than $B'B$, while $m+1$ of them exceed $B'B$, we can at any rate assert that the length of AB is greater than $n + \frac{m}{k}$ units but less than $n + \frac{m+1}{k}$; that is, we have measured *AB to the nearest kth part of the unit* less than its length. If we wish a still closer approximation, we have only to repeat the same process, taking a smaller fractional part of CD than we did before, that is, taking a *larger value of k. How* large a value of k we take (that is, *how* small a fractional part of CD we use) is a question that purely practical considerations answer. If the unit is the *foot,* often $k = 12$ would be sufficient; that is, we should be satisfied with measurement which is carried out to the nearest *inch.* We might, however, wish to know the length of our segment to the nearest *tenth of an inch,* in which case we should of course take $k = 120$, the unit being a *foot.*

The careful thinking through of this process will prepare the student to realize the truth of the following fact: *The length of any line segment* (measured by any unit whatever) *is always given, either exactly or approximately, as an integer or as the quotient of two integers* (that is, either as an *integer* or as a *fraction*).

The actual measurement of several concrete objects should be carried out by the student with special reference to the degree

of accuracy attained in cases where the segment measured does not seem to contain exactly the unit or the part of the unit chosen. Evidently a segment might happen to have such a length that *some* fractional part of the unit would be contained exactly in it, even though repeated trials might not show just *what part* is so contained. Thus, if a segment were exactly $3\frac{9}{13}$ units long, a measurement that gave the length to the nearest *third* of a unit would seem nearly exact, as $3\frac{2}{3}$ is only $\frac{1}{39}$ too small; and if we used *tenths* of a unit, we should get $3\frac{7}{10}$ as the length,—a result that differs from the actual length by only $\frac{1}{130}$. For most purposes this would be an entirely sufficient degree of accuracy, but we should never find an *exact* measure of the segment unless we divided the unit into precisely 13 parts (or some multiple of 13). Considerations of such examples of measurement as this one (and each student may easily work out several for himself) may very well suggest the conclusion that any segment could be measured in terms of a given unit and fractional parts of it, if we could only discover the correct fractional part to try. This conclusion, plausible though it seems, is, however, not correct; *there exist segments which can never be measured by a particular unit nor by any fractional part of it.* One example of such a segment is the diagonal of a square whose side is the unit.[1] Such segments are said to be *incommensurable* with the unit. The student will no doubt recall numerous examples. Of course these incommensurable segments can be measured *to any desired degree of approximation,* exactly as is done in the case of such a segment as the one mentioned just above, that was assumed $3\frac{9}{13}$ units long.

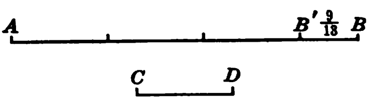


FIG. 3

3. Summarizing the results thus far obtained, we see that in measuring a line segment, after choosing a unit length, one of three things can happen: (1) the unit is contained exactly in the

[1] See Appendix A for a proof that the above statement is true of the diagonal of a square.

segment to be measured; in this case the length of the segment is an **integer**; or (2) the unit itself is not contained exactly in the segment, but some fractional part of the unit is; in this case the result of the measurement is a **fraction**, that is, the quotient of two integers; or, (3), neither the unit nor any fractional part of it is contained exactly in the segment; in this case the segment is *incommensurable*, and the length can be only approximately given as an integer or fraction. This length is, however, still spoken of as a *number*, but is called an **irrational number**, while the lengths of commensurable segments (that is, either integers or fractions) are called **rational numbers**.

The laws of elementary algebra include rules for working with irrational numbers, and with these rules the student is assumed to be familiar.[1]

4. Construction of segments of given length. The converse problem to *measurement* is *construction* of segments having a given ratio to the unit segment, that is, having a given length. The constructed segment will be said to *correspond* to the number given as its length, so that a segment of length 4 units will be said to correspond to the number 4.

(a) *Rational numbers.* Any segment whose length is a rational number can be constructed at once, because elementary geometry provides us with a method of constructing *any fractional part* of a unit segment. If the student has forgotten how to do this, let him review the method carefully. A brief statement of it is found in Appendix C.

(b) *Irrational numbers.* One would naturally take it for granted that, inasmuch as the segments whose lengths are irrational numbers are incommensurable, they would also be inconstructible; and in general this is true. But many such segments *can* be very easily constructed, namely, all those depending on *square roots* alone. The Pythagorean Theorem[2] gives us the means of doing

[1] See Appendix B for statement of these rules, with exercises.

[2] "The square on the hypotenuse of a right triangle is equal to the sum of the squares on the two legs." This very important theorem was discovered by the famous Greek philosopher and mathematician Pythagoras, in the sixth century B.C.

this for numbers like $\sqrt{2}$, $\sqrt{5}$, $\sqrt{\frac{1}{3}}$, $\sqrt{\frac{3}{7}}$, etc. For instance, $\sqrt{5}$ is the length of the hypotenuse of a right triangle whose legs are 1 and 2 in length; $\sqrt{\frac{3}{7}}$ is the length of one leg of a right triangle in which the hypotenuse is $\frac{5}{7}$ and the other leg $\frac{2}{7}$; $\sqrt{3}$ is the hypotenuse of a right triangle in which the legs are 1 and $\sqrt{2}$. To construct a segment whose length is the square root of the length of *any given segment*, see Appendix D, where the method is explained and illustrated. Finally, any segment whose length involves only rational combinations of square roots, that is, only additions, subtractions, multiplications, or divisions of square roots, can be constructed by combinations of the above-mentioned constructions,— for instance, such lengths as $1+\sqrt{2}$, $\dfrac{2}{3-\sqrt{5}}$, $\sqrt[4]{2}\left(=\sqrt{\sqrt{2}}\right)$, $\dfrac{2+\sqrt{3}}{2-\sqrt{3}}$.

EXERCISES

Construct accurately segments of the following lengths (using the same unit for all):

1. $\frac{3}{5}$.
2. $\frac{2}{11}$.
3. $\sqrt{7}$.
4. $1+\sqrt{2}$.
5. $\dfrac{\sqrt{5}-1}{2}$.
6. 1.7.
7. $\sqrt{3}$.
8. $\sqrt{\frac{4}{3}}$.
9. $\dfrac{1}{\sqrt{2}}$.
10. $\dfrac{3}{\sqrt{5}}$.
11. 3.9.
12. $3-\sqrt{3}$.
13. $\dfrac{5+\sqrt{6}}{5-\sqrt{6}}$.
14. $\dfrac{2-\sqrt{3}}{2+\sqrt{3}}$.
15. $\sqrt[4]{3}$.

5. This is as far as we can go in the construction of segments of given length, with the means of elementary geometry, — namely, straightedge and compass; for no cube root, fifth root, or other irrational number not reducible to square roots, can be so constructed.[1] But we are none the less instinctively certain that there *exists* a definite segment corresponding to any length, even to these inconstructible lengths. For example, if $AB = BC = 1$, we are convinced that there *exists*

FIG. 4

[1] This statement cannot be proved here, as it involves more advanced considerations than the student is yet prepared for.

a point X between B and C such that $AX = \sqrt[3]{2}$; and similarly for other such numbers. We can thus say that to any number, *rational or irrational,* corresponds a definite length, when once a unit has been chosen.

Negative Numbers

6. Directed segments. If we start from any point on a straight line to lay off a distance, it often makes a great difference *in which direction* we proceed to take the distance in question. It is accordingly useful to have some way of distinguishing, in such cases, which direction is to be taken. This is done by prefixing to each number used a label, or *sign,* + if the distance is to be taken in the one direction, and − if in the opposite direction. For instance, segments directed toward the *right* on a horizontal line may be given the sign + and called *positive* segments, while those directed toward the *left* will then be given the sign − and called *negative* segments. In Fig. 5, AB, AC, and BC are *positive* segments, while BA, CB, and CA are *negative.* The numbers expressing the *lengths* of such directed segments are then given the same signs as the segments themselves, and are spoken of as *positive* or *negative numbers,* as the case may be. Thus, if the measure of the segment AB is 5 units (Fig. 5), and if that of BC is 3 units, $AB = +5$, while $BA = -5$ and $CB = -3$. A line, such as ABC in this illustration, on which every segment has a direction as well as a length, is called a *directed line.*

Fig. 5

7. Theorems on directed segments. If A, B, and C are any points on a directed line, then

$$AB = -BA \tag{1}$$

and

$$AC + CB = AB. \tag{2}$$

The first theorem results from the definition of a directed segment. The second is self-evident when C is between A and B, and it is true *whatever* be the relative positions of the three points. Thus, in Fig. 5, $AC = +8$, $CB = -3$, and $AB = +5$; and it is true that $+8 + (-3) = +5$. This theorem (2) is of great

importance in our further work, and hence the student should verify it by numerous examples showing the various possible relative positions of the points A, B, and C. Indeed, the truth of (2) is obvious geometrically, since to pass from A to C, and then from C to B, will give the same displacement to the right (or left) as the single amount AB.

8. The student must be careful to notice that whenever we speak of *negative* numbers as the lengths of segments, the word "negative" refers merely to the *direction* in which the corresponding distance is measured; it is a *qualitative*, not a *quantitative*, adjective. It will be remembered that this is the case with all the illustrations of negative numbers that are given in text-books of elementary algebra; for example, if we call temperature above zero $+$, then temperature below zero is called $-$; if an amount of capital (asset) is considered $+$, then a liability is called $-$; and so on. In fact, whenever two sets of numbers can be associated with each other in such a way that one set gives *one kind* of number and the other set the *opposite kind*, we may call the numbers of the one set $+$ and those of the other set $-$. Whatever we agree shall be the meaning of the sign $+$, the sign $-$ means the exact opposite.

9. Correspondence between numbers and points on a line. Let us take any point O as starting-point, and (with any convenient unit) construct on the same straight line the segments $OA = +1$, $OB = +2$, $OC = +3$, $OA' = -1$, $OC' = -3$, etc. As B is the end-point of the segment determined by the length $+2$, we may say that the number $+2$ determines the point B, and, similarly, that the number $+3$ determines the point C, the number -1 the point A', the number -3 the point C', and so on. We see that in the same way any number determines some one definite point on the straight line (and, conversely, any point on the straight line determines one definite number), provided only that a starting-point, as O, and a unit segment, as OA, are assumed. This gives us a definite one-to-one

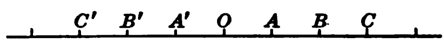


FIG. 6

correspondence between numbers and points on a (directed) straight line. Note that the number 0[1] corresponds to the point O.

10. The phrases "greater than" and "less than." Since, as we have seen, the word "negative" applied to numbers or distances is a qualitative, not a quantitative, adjective, it is evidently impossible to use the words "greater than" or "less than" in their ordinary quantitative sense when referring to negative numbers. A distance -3, for instance, is neither less than nor greater than a distance $+3$ in the ordinary sense, but, rather, an equal distance *in the opposite direction*. A meaning of these phrases that will be useful when applied to negative numbers is developed by the following considerations:

When all numbers are represented as points on a horizontal straight line, in the way that we have just illustrated, we find that, of two positive numbers, the point corresponding to the lesser is always *to the left* of that corresponding to the greater. Thus, the statement "$3 < 4$" is equivalent to the statement "the point 3 lies *to the left* of the point 4"; and so for all positive numbers. Now it is equally true that -1 lies to the left of 0, -2 to the left of -1, and so on; and so it is natural to express these facts also *by the same words that are used in the case of the positive numbers*: "-1 is less than 0," "-2 is less than -1," etc. In every case it should be remembered that the expression has no longer a quantitative, but a *directional*, meaning.

EXERCISES

1. Restate the last paragraph for the case where the numbers are represented as points on a vertical line; take the *upward* direction as positive.

2. Arrange in order of magnitude: $2, -3, \sqrt{5}, 4, -1.1, -\sqrt{10}, \sqrt{3}, -2, -2.8, 3.9, \sqrt{15}, -\sqrt{5}, \sqrt[3]{5}, \sqrt[3]{3}$.

3. State the laws of addition, subtraction, multiplication, and division with negative numbers.

[1] Many students are in the habit of thinking of "zero" as if it were not a number at all — sometimes even reading it "nothing." This is a mistake; zero *is* a number, in some respects indeed the most important number there is.

11. The system of coördinates. We have now seen that every number determines a certain point on a directed straight line, and that, conversely, to every point on a directed straight line corresponds a number, a starting-point (or zero-point) and a unit segment having been assumed. We now extend this correspondence between points and numbers to include all the points of a plane.

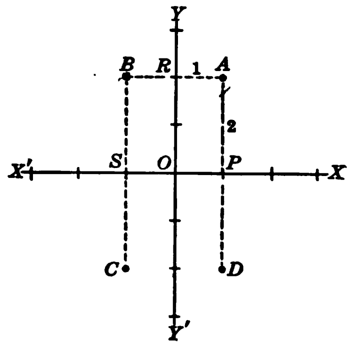


Fig. 7

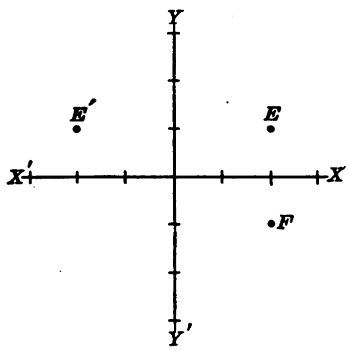


Fig. 8

Let XX' and YY' be two perpendicular lines intersecting at O, and suppose for convenience that XX' is horizontal. These lines will be referred to as the **X-axis** and the **Y-axis** respectively. Their intersection is called the **origin**. Then any point in the plane can be located exactly by giving its distances from the X-axis and from the Y-axis. Thus, the point A is $+1$ from the Y-axis and $+2$ from the X-axis, while the point B is -1 from the Y-axis and $+2$ from the X-axis. Similarly for any point in the plane. To locate the point it is sufficient to mention these two distances, *not forgetting the proper sign.* It will also save words if we agree once for all that the *horizontal* distance, or distance from the Y-axis, shall always be given first. Thus, merely mentioning the pair of numbers $(1, 2)$ is sufficient to locate the point A, $(-1, 2)$ locates the point B (Fig. 7), while $(2, 1)$ locates the point E (Fig. 8), and $(2, -1)$ the point F (Fig. 8). The student will be able to decide for himself what pair of numbers locates the point C, or the point D, in Fig. 7, or the point E' in Fig. 8.

It is thus clear that a pair of numbers can be considered as determining a definite point, and, conversely, that a given point

determines a definite pair of numbers, namely, its distances from the X- and Y-axes. The horizontal distance, or (directed) distance *from the Y-axis to the point,* is called the **abscissa** of the point; and the vertical distance, or (directed) distance *from the X-axis to the point,* is called the **ordinate** of the point. Thus, the abscissa of the point B (Fig. 7) is -1 (the directed segment RB or OS), and its ordinate is 2 (the directed segment SB or OR). Give the abscissa and the ordinate of the points E and F (Fig. 8). The abscissa and ordinate of a point are called its **coördinates.**

This system of connecting points with pairs of numbers was devised by Descartes, a great French mathematician and philosopher (1596–1650), and published by him in 1637 in a work called "La Géométrie," which is justly regarded as a milestone in the history of human thought. In honor of him the system of coördinates which has just been outlined is called the Cartesian coördinate system.

Locating points in a drawing by means of their coördinates is called *plotting* the points. For the sake of convenience in plotting points, squared paper is used, as it enables us to lay off any required distance both rapidly and accurately. In making all drawings a hard lead pencil with a fine point should be used, ink being reserved for certain lines which are meant to stand out prominently. In this first set of drawing exercises, draw the X- and Y-axes in ink. It is scarcely necessary to say that the greatest neatness and accuracy are absolutely indispensable in all problems involving drawing, as indeed in all mathematical work.

EXERCISES

1. Plot the points $(2, 1)$, $(2, -1)$, $(-2, 1)$, and $(-2, -1)$. What kind of figure do they form?

2. Plot the points $(2, 0)$, $(-1, 0)$, $(\sqrt{2}, 0)$, $(5, 0)$, $(-4\frac{1}{2}, 0)$, $(-\sqrt{2}, 0)$, $(3\frac{2}{3}, 0)$. On what line are they all found?

3. On what line are all the following points located? $(1, 1)$, $(-2, -2)$, $(3, 3)$, $(-\frac{1}{2}, -\frac{1}{2})$, $(5, 5)$, $(2.1, 2.1)$, $(-\sqrt{2}, -\sqrt{2})$, $(1+\sqrt{2}, 1+\sqrt{2})$, $(-\frac{4}{3}, -\frac{4}{3})$.

4. Plot the points $(-4, -3)$, $(3, -4)$, $(4, -3)$, $(3, 4)$, $(4, 3)$, $(-3, 4)$. Prove that they all lie on a circle, and give the coördinates of other points on the same circle.

5. Plot the points $(0, 0)$, $(\sqrt{2}, \sqrt{2})$, $(-\sqrt{2}, \sqrt{2})$. What kind of figure do they form?

6. An equilateral triangle whose side is 4 units is placed with one vertex at the origin and with the opposite side perpendicular to the X-axis. Find the coördinates of the other two vertices. (Two solutions.)

7. A square whose side is 2 units is placed with one vertex at the origin and with a diagonal lying along the X-axis. Find the coördinates of the other three vertices and of the center. (Two solutions.)

8. A regular hexagon whose side is the unit is placed with one vertex at the origin and with its center on the X-axis. Find the coördinates of the other 5 vertices, and the area of the hexagon. (Two solutions for the vertices.)

9. Plot the points $(3, 1)$, $(0, 4)$, and $(3, 4)$. What kind of figure do they form? Find the lengths of its sides, its area, and the coördinates of the center of the circumcircle.[1]

10. How far is the point $(3, 4)$ from the origin in a straight line? the point $(1, 5)$? the point $(-1, -\frac{3}{4})$? the point (a, b)?

11. Plot the points $(3\frac{4}{5}, 3\frac{2}{5})$, $(\frac{1}{5}, 3\frac{2}{5})$, and $(3\frac{4}{5}, -1\frac{2}{5})$, and find the center and radius of the circle through the three.

12. What are the coördinates of the points P, R, and S in Fig. 7? What are the coördinates of the origin?

13. What are the coördinates of the point halfway between the origin and the point $(4, 4)$? of the point halfway between the origin and the point $(3, 6)$?

14. Plot the points $(2, 1)$ and $(4, 5)$. What are the coördinates of the point midway between them?

15. Answer the same question for the points $(1, 3)$ and $(-3, 1)$; for the points $(1, 0)$ and $(0, 1)$; for the points $(3, -2)$ and $(-5, 4)$.

[1] Circumcircle: a circle passing through the vertices of a triangle or of any polygon.

16. Let (x_1, y_1)[1] and (x_2, y_2) be any two points. Find the coördinates of the mid-point of the segment joining them.

Ans. $\dfrac{x_1 + x_2}{2}, \dfrac{y_1 + y_2}{2}$.

17. Find the distance between the points $(-1, 2)$ and $(-4, 6)$.

Solution. If A (Fig. 9) represents the point $(-1, 2)$ and B the point $(-4, 6)$, and if we draw BD and AE perpendicular to the X-axis, and AC perpendicular to BD, we have formed a right triangle ABC whose hypotenuse is the required distance AB. But the lengths of AC and CB can easily be found, as follows:

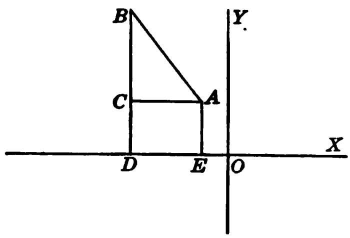


FIG. 9

$$OE = -1$$

and $$OD = -4.$$

Therefore $$ED = AC = -3.$$

Likewise $$DB = 6$$

and $$DC = 2.$$

Therefore $$CB = 4.$$

By the Pythagorean Theorem, $$AB^2 = AC^2 + CB^2.$$

Therefore $$AB^2 = 9 + 16 = 25.$$

Hence $$AB = 5.$$

18. Find the distance between the points $(3, -1)$ and $(4, -\frac{1}{4})$; between the points $(6, 1)$ and $(-6, 6)$.

19. Find the distance from $(0, 3)$ to $(4, 2)$; from $(1, 1)$ to $(-2, -3)$; from $(3, 5)$ to $(-4, 1)$.

20. Find the distance between the points (x_1, y_1) and (x_2, y_2).

Ans. $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$. This result is easily established when the given points (x_1, y_1) and (x_2, y_2) both have *positive* coördinates, but the student should show that it is true for *all* positions of the two points. The theorems in § 7, p. 6, will be found useful for this purpose.

12. Application of coördinates to some problems of elementary geometry. This method of representing points by pairs of numbers is very useful because it gives us a means of handling many

[1] Read "x-one, y-one," meaning "the first x," and so on. Such subscripts are often very convenient, and should not be confused with exponents.

geometric problems in an algebraic way. The significance of this statement will become much more apparent as we proceed farther in our work, but the simple examples which follow will provide at least a basis for appreciation of it.

Example 1. Let us take the well-known problem of elementary geometry, Prove that the diagonals of a rectangle are equal. In such problems, where the object is to prove geometric theorems by the aid of coördinates, the secret of a simple solution lies in a wise *choice of the coördinate axes.* We are at liberty to choose them in any position we please with reference to any figure already given. In this case we choose as X- and Y-axes two adjacent sides of the given rectangle, and suppose the length of OP (Fig. 10) is a and that of OR is b. Then the coördinates of O are $(0, 0)$, those of P are $(a, 0)$, those of R are $(0, b)$, and those of Q are (a, b). We now apply Ex. 20, above, which gives us

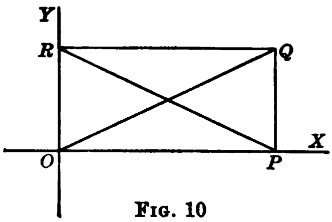


FIG. 10

$$RP = \sqrt{(a-0)^2 + (0-b)^2} = \sqrt{a^2+b^2}$$

and
$$OQ = \sqrt{(a-0)^2 + (b-0)^2} = \sqrt{a^2+b^2}.$$

Therefore $RP = OQ$. Q.E.D.

Example 2. Another well-known theorem of elementary geometry which can easily be proved by the use of coördinates is, The diagonals of a parallelogram bisect each other.

Proof. Let $OPQR$ (Fig. 11) be the parallelogram. Choose one side OP as the X-axis, and O as the origin. The coördinates of P, the other vertex on the X-axis, may then be represented by $(a, 0)$, those of R by (b, c), and those of Q by $(a + b, c)$. (For $MR = b$ and $RQ = OP = a$. $\therefore MQ = a + b$, the abscissa of Q.)

Then the coördinates of the mid-points of the diagonals can be found by Ex. 16, above. For the diagonal RP the mid-point is thus $\left(\frac{a+b}{2}, \frac{c}{2}\right)$, and for the diagonal OQ it is $\left(\frac{a+b}{2}, \frac{c}{2}\right)$, that is, the same point. Since their mid-points coincide, the diagonals bisect each other. Q.E.D.

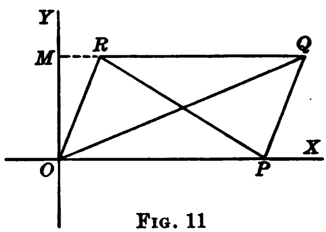


FIG. 11

EXERCISES

Prove the following theorems by means of coördinates:

1. The line joining the vertex of any right triangle to the mid-point of the hypotenuse is equal to half the hypotenuse.

2. The line joining the middle points of two sides of a triangle is equal to half the third side.

3. The distance between the middle points of the nonparallel sides of a trapezoid is equal to half the sum of the parallel sides.

4. In any quadrilateral the lines joining the middle points of the opposite sides and the line joining the middle points of the diagonals meet in a point and bisect each other.

5. If the lines joining two vertices of a triangle to the middle points of the opposite sides are equal, the triangle is isosceles.

13. Point that divides a segment in a given ratio. Ex. 16, p. 12, enables us to find the coördinates of the mid-point of any

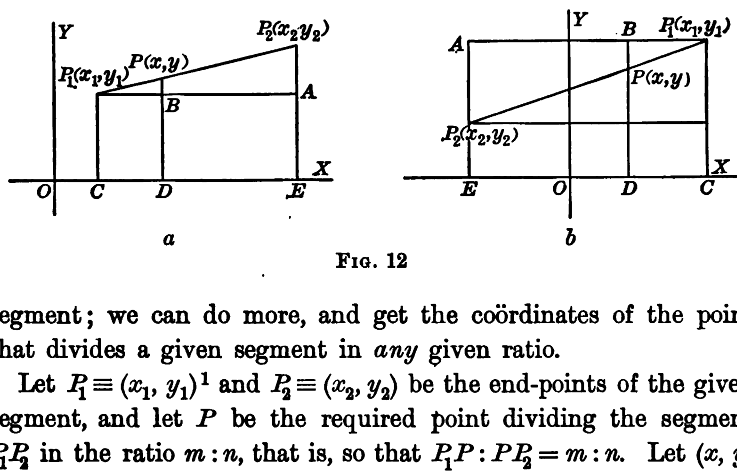


Fig. 12

segment; we can do more, and get the coördinates of the point that divides a given segment in *any* given ratio.

Let $P_1 \equiv (x_1, y_1)$[1] and $P_2 \equiv (x_2, y_2)$ be the end-points of the given segment, and let P be the required point dividing the segment P_1P_2 in the ratio $m : n$, that is, so that $P_1P : PP_2 = m : n$. Let (x, y) be the coördinates of P. Then, if P_1C, PD, and P_2E are perpendiculars from P_1, P, and P_2 respectively, upon the X-axis, OC equals the abscissa of P_1 (that is, x_1), $OE = x_2$, and $OD = x$. Form the right triangle P_1AP_2 by drawing through P_1 the line

[1] Read "P_1, which is the point (x_1, y_1)" or "P_1, whose coördinates are (x_1, y_1)."

P_1A parallel to the X-axis, meeting PD in B and P_2E in A. Then $P_1A = CE = x_2 - x_1$ (notice that this is true even if P_2 is to the left of P_1, as in Fig. 12 *b*, since $x_2 - x_1$ is then *negative*), $P_1B = CD = x - x_1$, and $BA = DE = x_2 - x$.

By hypothesis $P_1P : PP_2 = m : n$, and since the triangles P_1BP and P_1AP_2 are similar (why?), we have $P_1P : PP_2 = P_1B : BA$. But this last ratio is equal to $\dfrac{x - x_1}{x_2 - x}$; hence

$$\frac{x - x_1}{x_2 - x} = \frac{m}{n}. \tag{1}$$

Solving (1) for x,

$$\mathbf{x = \frac{mx_2 + nx_1}{m + n}}. \tag{1'}$$

In exactly the same way, drawing PF perpendicular to P_2A (not shown in Fig. 12), we find $AF = BP = y - y_1$, $FP_2 = y_2 - y$, and thus

$$\frac{y - y_1}{y_2 - y} = \frac{m}{n}. \tag{2}$$

Solving (2) for y,

$$\mathbf{y = \frac{my_2 + ny_1}{m + n}}. \tag{2'}$$

Notice that these results (1′) and (2′) reduce to the results of Ex. 16, p. 12, when $m = n$. Observe also that m and n are not necessarily the *lengths* of the segments P_1P and PP_2, but are merely *numbers in the same ratio as those lengths*, m corresponding to the segment P_1P (that is, to the segment nearest P_1), while n corresponds to the segment PP_2 (that is, to the segment nearest P_2).

Example 1. Find the coördinates of the point which divides the segment from (1, 3) to (6, 2) in the ratio 2 : 3.

Solution. Here

$$m = 2,\ n = 3,$$
$$x_1 = 1,\ y_1 = 3,$$
$$x_2 = 6,\ y_2 = 2.$$

Substituting these values in (1′),

$$x = \frac{2 \cdot 6 + 3 \cdot 1}{2 + 3} = 3,$$

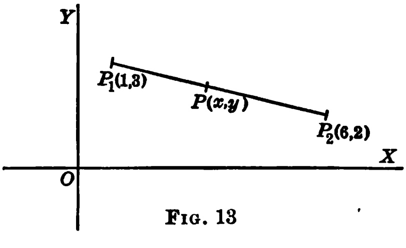


FIG. 13

and from (2′),

$$y = \frac{2 \cdot 2 + 3 \cdot 3}{2 + 3} = \frac{13}{5}.$$

Therefore

$$P \equiv (3, \tfrac{13}{5}).$$

Example 2. Find the coördinates of the point $\frac{1}{3}$ of the way from $(-4, 2)$ to $(2, -1)$.

Solution. The point P being $\frac{1}{3}$ of the way from P_1 to P_2, the segment $P_1P = \frac{1}{3}P_1P_2$, and hence $PP_2 = \frac{2}{3}P_1P_2$, so that $P_1P : PP_2 = 1:2$. (A rather natural although careless mistake that is often made in problems of this kind is to say $m = 1$, $n = 3$, since the point P is to be $\frac{1}{3}$ of the way from P_1 to P_2.) The student can now finish the work for himself. The result is $P \equiv (-2, 1)$.

Example 3. Prove that the medians of any triangle meet in a point $\frac{2}{3}$ of the way from a vertex to the mid-point of the opposite side.

Proof. Let $A \equiv (x_1, y_1)$, $B \equiv (x_2, y_2)$, and $C \equiv (x_3, y_3)$ be the vertices of the triangle. Then the coördinates of M, the mid-point of AB, are $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$; those of N, the mid-point of BC, are $\left(\frac{x_2 + x_3}{2}, \frac{y_2 + y_3}{2}\right)$; and those of Q, the mid-point of CA, are $\left(\frac{x_3 + x_1}{2}, \frac{y_3 + y_1}{2}\right)$.

Next, the coördinates of the point $\frac{2}{3}$ of the way from C to M are as follows $(m = 2,\ n = 1)$:

$$x = \frac{2 \cdot \frac{x_1 + x_2}{2} + 1 \cdot x_3}{2 + 1} = \frac{x_1 + x_2 + x_3}{3},$$

$$y = \frac{2 \cdot \frac{y_1 + y_2}{2} + 1 \cdot y_3}{2 + 1} = \frac{y_1 + y_2 + y_3}{3}.$$

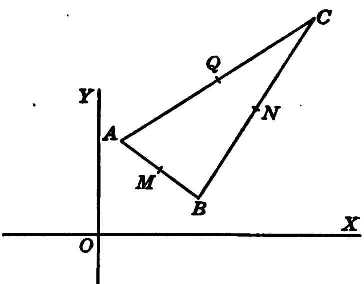


FIG. 14

Working out the coördinates of the point $\frac{2}{3}$ of the way from A to N, also of the point $\frac{2}{3}$ of the way from B to Q, we get in each case the same pair of coördinates. This proves the theorem.

EXERCISES

1. Find the coördinates of the point which divides the segment from $(1, 3)$ to $(-2, 6)$ in the ratio $1:2$; in the ratio $2:1$.

2. Find the coördinates of the point $\frac{2}{3}$ of the way from $(-2, -3)$ to $(7, 3)$; $\frac{2}{3}$ of the way from $(\frac{1}{2}, 3\frac{2}{3})$ to $(-1, -\frac{1}{3})$.

3. Prove that in any parallelogram $ABCD$, if M is the middle point of the side AB, the line MD and the diagonal AC trisect each other.

4. Find the coördinates of the point that divides the segment from (0, 3) to (2, 5) *externally* in the ratio 3 : 2.

HINT. Here the point P is on the line P_1P_2 *produced*, and since $P_1P : PP_2$ equals numerically $\frac{3}{2}$, P_1P must be *greater* than PP_2; that is, P must be nearer to P_2 than to P_1, or, in other words, it is beyond the point P_2. P_1P and PP_2 now being measured in opposite directions, we may think of the ratio $m : n$ as being *negative*. With this understanding it will be found that the reasoning by which the formulas (1) and (2) were obtained is still valid, and they may be used in these cases also. Here $m = 3$, $n = -2$ (or else $m = -3$, $n = 2$), so that we have

$$x = \frac{3 \cdot 2 - 2 \cdot 0}{3 - 2} = 6, \qquad y = \frac{3 \cdot 5 - 2 \cdot 3}{3 - 2} = 9.$$

Hence the required point is (6, 9).

5. Find the coördinates of the point that divides the segment from $(-2, -3)$ to $(3, -5)$ in the ratio $-2 : 3$; in the ratio $-1 : 2$; in the ratio $-2 : 1$.

6. A line AB is produced to C so that $AC = 3\,BC$. Find the coördinates of C if $A \equiv (2, 0)$ and $B \equiv (-3, 1)$.

7. Find the coördinates of the point that divides in the ratio λ the segment from $(1, 2)$ to $(-1, 3)$; from (x_1, y_1) to (x_2, y_2).

8. Find the point of intersection of the medians of the triangle whose vertices are $(2, 3)$, $(4, -5)$, and $(3, -6)$. (Do not use the *result* of Ex. 3, p. 16, but follow the same method, using the special numbers of this problem.)

9. In what ratio does the point $(2, 3)$ divide the segment from $(-2, -3)$ to $(4, 6)$?

HINT. Use equations (1) and (2), p. 15.

10. In what ratio is the segment from $(-2, 1)$ to $(3, -9)$ divided by the point $(1, -5)$?

11. If $A \equiv (2, -1)$ and $B \equiv (5, \frac{1}{2})$, in what ratio is AB divided by the point $C \equiv (4, 0)$?

12. Using the same segment AB as in the preceding problem, show that, for the point $D \equiv (-5, -4)$, $\dfrac{x - x_1}{x_2 - x} = -\dfrac{7}{10}$, while $\dfrac{y - y_1}{y_2 - y} = -\dfrac{2}{3}$, a *different ratio*. What conclusion can you draw from this peculiarity?

13. Test as in the preceding problem the three points $A \equiv (3, 2)$, $B \equiv (-1, -10)$, and $C \equiv (0, -8)$; the points $(5, -1)$, $(2, 2)$, and $(-1, 5)$.

14. Formulate the results of Exs. 12 and 13 in the form of a theorem that gives a test as to *whether three points A, B, and C are in the same straight line or not.*

15. Given the three points $A \equiv (2, 1)$, $B \equiv (-1, 3)$, and $C \equiv (1, -2)$, find the coördinates of D, the fourth vertex of the parallelogram determined by A, B, and C. (Three solutions.)

16. Proceed as in Ex. 15 for the three points $(0, 5)$, $(7, 3)$, and $(-2, -3)$.

17. Prove that the sum of the squares of the medians of any triangle equals three fourths the sum of the squares of the sides.

CHAPTER II

FUNCTIONS AND THEIR GRAPHS

14. In the preceding chapter we have seen how to associate with every segment a number, — its length; and by means of the system of coördinates we have associated with every point in the plane a pair of numbers. We applied this to the proof of a few theorems of elementary geometry. In this chapter we shall consider not merely *fixed* points, as heretofore, but *variable* points, that is, points that are free to occupy any number of positions, subject to certain conditions.

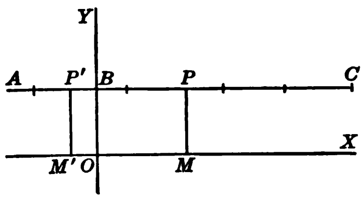


FIG. 15

For example, let us say that a point is free to move so as to be always at the distance $+2$ from the X-axis. There are evidently an unlimited number of such points, but they will all be found on one straight line, the line ABC, parallel to the X-axis and 2 units above it. We use the expression, The *locus* of points at the distance $+2$ from the X-axis is the line parallel to the X-axis and 2 units above it. Furthermore, to specify that the point shall be at the distance 2 from the X-axis is the same thing as saying that its ordinate shall equal 2; or, writing y for the ordinate of any one of the points in question, the original condition is equivalent to the statement that $y = 2$ (or $y - 2 = 0$). We may say, then, that *the locus of the equation* $y = 2$ (or $y - 2 = 0$) *is the straight line parallel to the X-axis and 2 units above it.* We call the line also the *graphical representation*, or simply the *graph*, of the equation.

15. As another example, let us make the following condition: A point moves so as to be always twice as far from the X-axis as from the Y-axis. Here again we know from elementary geometry

that the locus of these points is a straight line $P'OP$, for if P is twice as far from the X-axis as from the Y-axis (that is, if $QP = 2 \cdot RP = 2 \cdot OQ$), then the same thing is true for *any* point on the line OP, and for no other points.[1] Since the given condition is equivalent to the condition that the ordinate of the point shall equal twice its abscissa, we can express this condition in the form of an equation, thus: $y = 2x$, where x represents the abscissa of the moving point, and y its ordinate.

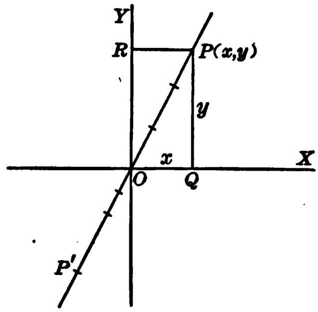


FIG. 16

16. This example illustrates the use of the general numbers (x, y) to represent the coördinates of a *variable*, or *moving*, point, — a notation which is very convenient and which will be frequently employed throughout this book. The quantities that we have to deal with in physics, engineering, astronomy, and indeed in all applications of mathematics, are largely *variable*, and hence it is necessary to learn to use and understand symbols that represent such variable quantities. To represent *constant* quantities we shall use the earlier letters of the alphabet or letters with subscripts, as x_1, y_1, etc.

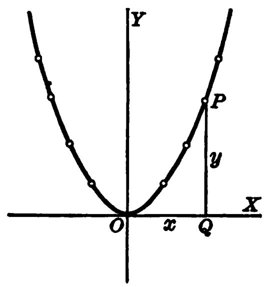


FIG. 17

17. As a third example let us make the following condition: A point moves so that its ordinate is always equal to the square of its abscissa. If the coördinates of this variable point are represented by (x, y), then evidently the equation of the locus is $y = x^2$; but elementary geometry gives us no information as to just *what* this locus is, that is, just how the point can be located if it satisfies this condition. We can, however, get a good idea of the locus by *starting with the equation* $y = x^2$

[1] For the present this assertion may be accepted without proof. The proof is given on page 92, where this example is taken up again more in detail.

itself and finding a number of pairs of values of x and y which satisfy the equation. If we then plot the points having such values (x, y) as their coördinates, we shall get a series of points satisfying the given condition; that is, we shall get points on the locus. Thus, when $x=1$, $y=1$; when $x=2$, $y=4$; when $x=\frac{1}{2}$, $y=\frac{1}{4}$; when $x=0$, $y=0$; etc. Hence $(1, 1)$, $(2, 4)$, $(\frac{1}{2}, \frac{1}{4})$, and $(0, 0)$ are points on the locus. We may conveniently arrange these pairs of values in the form of a table, thus:

x	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2	-1	-2	$-\frac{1}{2}$	$-\frac{3}{2}$
y	0	$\frac{1}{4}$	1	$\frac{9}{4}$	4	1	4	$\frac{1}{4}$	$\frac{9}{4}$

etc.

We now plot the points carefully, and when enough points have been located, it will be possible to see that they suggest a curved line, concave upward, symmetrical with respect to the Y-axis, and with its lowest point at the origin. This curve should now be carefully drawn on a large scale, and we can then infer that we have a fairly accurate drawing of the locus of the equation $y=x^2$. The curve is called a *parabola.* (It must not be overlooked, however, that we cannot *prove* that all points on the curve satisfy the condition $y=x^2$; but we *assume* this to be the case, and if the drawing has been made accurately, this assumption will be in fact approximately correct.)

18. As a fourth example, let us suppose that a point (x, y) moves so that the following relation between x and y always holds: $y=x^3-2x^2-x+4$. As before, we proceed to compute a table of values of x and y that satisfy this equation. The result may be exhibited thus:

x	0	1	2	3	-1	-2	-3	$\frac{1}{2}$	$\frac{3}{2}$	$-\frac{1}{2}$
y	4	2	2	10	2	-10	-38	$3\frac{1}{8}$	$1\frac{3}{8}$	$3\frac{7}{8}$

Plotting the points given by this table, and connecting them by a smooth curve, as in Fig. 18, we get approximately the graphical representation of the equation given.

In the following exercises a point is to be understood as varying subject to the condition given each time in the form of an equation between x and y. Draw the graph of the equation carefully, determining the *exact* locus in Exs. 1–10, and a very close approximation in the others, as was done in the last two examples above. Also, in the first 12 exercises state in words the condition which the equation gives in symbols: for example, in No. 1, "If a point moves so that its abscissa equals 2, what is its locus?"

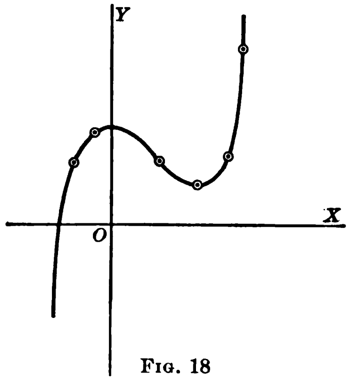


FIG. 18

EXERCISES

1. $x = 2$.
2. $y = -3$.
3. $x = 0$.
4. $y - 5 = 0$.
5. $y = 3x$.
6. $x = -2$.
7. $y = 0$.
8. $y = -2x$.
9. $y = -x$, and $y = -x + 2$, in the same figure.
10. $y = \frac{1}{2}x$, and $y = \frac{1}{2}x - 3$, in the same figure.
11. $y = 2x - 1$.
12. $y = 3x + 4$.
13. $y = -\frac{2}{3}x + 1$.
14. $y = 2x^2 - 2$.
15. $y = -3x^2 - 5$.
16. $y = x^2 - x$.
17. $y = x^2 - 2x$.
18. $y = -x^2 - x$.
19. $y^2 = -x + 3x$.
20. $y = 2x^2 - 3x$.
21. $y = 2x^2 - x + 3$.
22. $y = -x^2 - x + 5$.
23. $y = -2x^2 + x + 1$.
24. $y = x^3$.
25. $y = x^3 - 1$.
26. $y = x^3 - x$.
27. $y = x^3 + x$.
28. $y = x^3 - x^2$.
29. $y = -x^3 + 3x^2$.
30. $y = x^3 - 3x^2 + 1$.
31. $y = -2x^3 - 6x$.

32. $y = x^3 + x^2 + x + 1.$

33. $y = x^3 - x^2 + x - 1.$

34. $y = x^3 - x^2 - x - 1.$

35. $y = -x^3 + 2x^2 + 3.$

36. $y = -2x^3 + x^2 - x + 2.$

37. $y = -x^3 - x^2 - x - 3.$

19. Definition of function. In all of these later exercises, pairs of values (x, y) that would satisfy the equation given were determined by taking a set of values of x and then using the equation to compute the corresponding values of y. Whenever a pair of (variable) numbers x and y are related in this way, that is, in such a way that to a value of the one corresponds a definite value of the other, the second is said to be a **function** of the first, or the relation is called a **functional** relation. The symbol for this relation between x and y is $y = f(x)$, read "y is a function of x" or "$y = f$ of x." For example, if $y = 2x$, y is a function of x, because to a given value of x corresponds a definite value of y; or, again, if $y = x^3 - 2x^2 - x + 4$, y is a function of x. In these examples x is called the *independent* variable, and y, or the *function of* x, is called the *dependent* variable, because its value *depends* upon that of x.

20. Importance of functional relation. The idea of this *functional relation* between two variable numbers is of very far-reaching importance, because it can be applied to any pair of quantities that are representable by numbers if one of these numbers is determined by the value of the other. For instance, the temperature at any place is a function of the *time*, because at any definite *time* there is a definite *temperature* at that place; or, again, if a man walks at the rate of 3 miles per hour, the *distance* he has gone is a function of the *time* he has walked; or, again, the population of a city is a function of the time (date) when the estimate or count is made. Consideration of the wide variety of such possible illustrations will make clear the very great importance of a study of various kinds of functional relations.

21. For such a study the graphical representation just illustrated is a most valuable aid, because it gives us a vivid ("graphic") picture of the functional relation, thus assisting materially in forming a clear

idea of the nature of that relation. This may be seen in the example mentioned above of the variation of temperature at a certain place; suppose the weather observer had made the following record:

TIME	TEMPERATURE
7 A.M.	51°
8 A.M.	55°
9 A.M.	60°
10 A.M.	62°
11 A.M.	70°
12 M.	72°
1 P.M.	72°
2 P.M.	73°
3 P.M.	71°
4 P.M.	66°
5 P.M.	61°
6 P.M.	58°

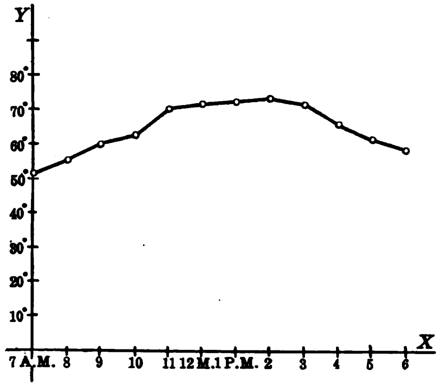


FIG. 19

Since the first observation was taken at 7 A.M., we represent that time as the origin and construct the *times* as abscissas and the *temperatures* as ordinates, thus getting the above figure. A single glance at the figure evidently gives us a clearer and more comprehensive idea of the variation of the temperature on that particular day than the table of values does.

22. To take a more elaborate illustration of this use of the graphical representation, we find in the *United States Statistical Abstract* for 1915 the following table showing the progress of shipbuilding in the United States from 1900 to 1915:

YEAR	TONNAGE BUILT	YEAR	TONNAGE BUILT
1900	393,790	1908	614,216
1901	483,489	1909	238,090
1902	468,831	1910	342,068
1903	436,152	1911	291,162
1904	378,542	1912	232,669
1905	330,316	1913	346,155
1906	418,745	1914	316,250
1907	471,332	1915	225,122

Here the tonnage built is a function of the time. Plotting the number of the year as abscissa (starting of course with 1900 at the origin) and the corresponding tonnage as ordinate, we get a figure like the one adjoining. Let the student complete the figure, using as large a scale as possible. Now a glance at the broken line joining these points gives a much clearer idea of the variation in the ship-building during these years than does the mere table from which the diagram was constructed.

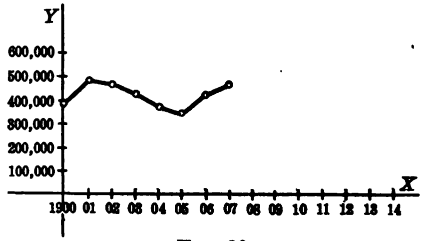


FIG. 20

23. In drawing these graphical representations of statistical tables it is not necessary to try to draw a smooth curve joining the points located, since we have no possible way of telling how the curve should look, *between* the located points; hence we draw only a broken line, that is, join each point to the next following by a *straight line.* In the graphical representation of *equations,* however, as in the exercises on page 22, it would not be right to do this, because we *can* locate points just as close together as we need, thus determining the shape of the curve to any desired degree of accuracy. For instance, in the example of § 18 we had (among others) the points (0, 4) and (−1, 2), which are so close together that it would be natural to connect them by a straight line (just as we *should* do if there were no way of telling how the curve really lies between these points); but by taking $x = -\frac{1}{2}$ we found $y = 3\frac{7}{8}$, which shows that the straight line joining (0, 4) and (−1, 2) would be decidedly incorrect. Between $x = 1$ and $x = 2$ it would be natural to make the same mistake again, and join (1, 2) and (2, 2) by a horizontal line; but the point $(\frac{3}{2}, 1\frac{5}{8})$ corrects this. The detailed study of the graphical representation of functions in this way is made much simpler and more practical by the use of a chapter in mathematical analysis which is called the Differential Calculus. Until the use of that very powerful method has been learned, however, the student

must be content to plot accurately a sufficient number of points so that the form of the curve is evident.

Summarizing, the graphical representation of functions given by mathematical *equations* can be carried out to any required degree of accuracy, while in the case of functions given by tables of observations or statistics we cannot, of course, locate more than the number of points given in the tables.

EXERCISES

1. The *Statistical Abstract* for 1915 gives the following figures for the values of exports and imports of merchandise for the years 1900–1915:

Year	Exports	Imports	Year	Exports	Imports
1900	1,394,483,082	849,941,184	1908	1,860,773,346	1,194,341,792
1901	1,487,764,991	823,172,165	1909	1,663,011,104	1,311,920,224
1902	1,381,719,401	903,320,948	1910	1,744,984,720	1,556,947,430
1903	1,420,141,679	1,025,719,237	1911	2,049,320,199	1,527,226,105
1904	1,460,827,271	991,087,371	1912	2,204,322,409	1,653,264,934
1905	1,518,561,666	1,117,513,071	1913	2,465,884,149	1,813,008,234
1906	1,743,864,500	1,226,562,446	1914	2,364,579,148	1,893,925,657
1907	1,880,851,078	1,434,421,425	1915	2,768,589,340	1,674,169,740

Make a graphical representation of these statistics.

2. We find in the *Statistical Abstract* that the number of immigrants admitted to the United States during each of the years from 1892 to 1915 was as follows:

Year	Number Admitted	Year	Number Admitted	Year	Number Admitted
1892	623,084	1900	448,572	1908	782,870
1893	502,917	1901	487,918	1909	751,786
1894	314,467	1902	648,743	1910	1,041,570
1895	279,948	1903	857,046	1911	878,587
1896	343,267	1904	812,870	1912	838,172
1897	230,832	1905	1,026,499	1913	1,197,892
1898	229,299	1906	1,100,735	1914	1,218,480
1899	311,715	1907	1,285,349	1915	326,700

Draw a graphical representation of these facts.

3. The number of persons killed in railway accidents in the United States during each of the years from 1892 to 1910 was as follows:

YEAR	NUMBER OF PERSONS	YEAR	NUMBER OF PERSONS
1892	2554	1902	2969
1893	2727	1903	3606
1894	1823	1904	3632
1895	1811	1905	3361
1896	1861	1906	3929
1897	1693	1907	4534
1898	1958	1908	3405
1899	2210	1909	2610
1900	2550	1910	3382
1901	2675		

Make a graphical representation of these statistics.

4. Find statistics for the growth of the population of the United States since 1790, and draw the graphical representation of these figures.

5. Answer the same question for three or four of the states, from the time they were admitted to the Union to the present.

6. Make a graphical representation of the amount of $1000 at simple interest for one year, as a function of the rate per cent (from 1% to 10%).

7. Answer the same question for the amount of $1000 at 5% as a function of the time (from 1 yr. to 10 yr.).

NOTE. Ample material for further work in statistical graphs may be found in the government census reports, crop reports, and in the financial journals, etc.

CHAPTER III

APPLICATION OF GRAPHICAL REPRESENTATION TO ELEMENTARY ALGEBRA

24. In the preceding chapter we saw how to use the coördinate system to draw the graphs of many simple functional relations between x and y; and we observed that such graphical representation of equations in the form $y = f(x)$ is of very wide application in various fields. In this chapter we shall use this geometrical work of drawing graphs for the purpose of throwing new light upon certain problems of elementary algebra. As a preliminary step it will be found useful to make a simple classification of functions according to their *degree.*

25. Degree of a function. The expressions $2x-3$, $-5x+2$, $3x$, $\frac{1}{2}x-1$, $\frac{x}{5}+7$ are all functions of x, according to the definition of the word "function," for the value of any one of these quantities is determined by the value of x. In each of them the variable number x occurs to no higher power than the first. Functions in which this is the case are said to be "of the first degree" or "linear" in x. A *general* form of such functions of x would be $ax+b$, where a and b are *general numbers*, that is, may have any value we please.

A function which contains x^2 but no higher power of x is said to be "of the second degree" or "quadratic" in x. A general form for it would be ax^2+bx+c. A function which contains x^3 but no higher power of x is said to be "of the third degree" or "cubic" in x, a general form being ax^3+bx^2+cx+d. In the list of problems on page 22 we had examples of all these three kinds of functions, — linear, quadratic, and cubic. In the same way it is possible to write down functions of the fourth, fifth, $\cdots$ degree, but we shall seldom need to go beyond the third degree in this book.

26. Linear equations. An equation of the form

$$ax + by = c \tag{1}$$

is called a *linear equation* in the variables x and y. If this equation be solved for y, a linear function is obtained: $y = \frac{c}{b} - \frac{a}{b} \cdot x$. For example, if we had $2x + 3y = -1$, then $y = -\frac{1}{3} - \frac{2}{3}x$, which is a linear function of x. Let the student write down, at random, three or four equations of the form (1), choosing any numbers whatever for a, b, and c, and then make a careful graph of the function in each case. Each graph will turn out to be a straight line. Later on we shall be able to *prove* that this is necessarily the case, but for the present we merely *assume* it to be a fact, — that is, we make, as yet without proof, the following assumption:

The graph of every equation of the first degree in x and y is a straight line.

This is the reason why such an equation or functional relation is called "linear." Accordingly, in making the graph of such an equation it is sufficient to plot *two* points whose coördinates satisfy the equation, and then the straight line joining these two points will be the graph of the equation. (In practice a third point should also be plotted, as a check; if it is not exactly on the graph, a mistake has been made.) For the two points it is often most convenient to choose the points where the line crosses the X-axis and the Y-axis. For the point of intersection with the X-axis the ordinate equals 0 (since the ordinate of any point on the X-axis equals 0); hence let $y = 0$ in the equation of the line, and solve for the value of x. Similarly, to get the point of intersection with the Y-axis, let $x = 0$ in the equation. The (directed) distances from the origin to these points of intersection with the X- and Y-axes are called the **X- and Y-intercepts** of the line. In general, if in the equation of any locus we let $x = 0$, we obtain the value of the ordinate of the point where the locus meets the Y-axis; and if we let $y = 0$, we get the abscissa of the point where the locus meets the X-axis. These two points are especially useful in the practical work of drawing graphs.

27. Simultaneous linear equations. It is a familiar and simple problem of elementary algebra to find a pair of values (x, y) that will satisfy *each* of two linear equations in the variables x and y. We can now solve this problem graphically, that is, geometrically, in a very simple manner. Let the equations be

$$ax + by = c \tag{1}$$

and

$$a'x + b'y = c'. \tag{2}$$

We draw in the same figure the graphs of the two equations; then the graph of equation (1) contains all points whose coördinates (x, y) satisfy that equation (for this is the *definition* of the graph of an equation), and the graph of equation (2) contains all points whose coördinates (x, y) satisfy that equation. Hence any point whose coördinates (x, y) satisfy *both equations at once* must be found on *both graphs*, that is, must be their point of intersection. The solution of the problem can accordingly be obtained, at least approximately, by a glance at the figure; and since the graphs of linear equations are straight lines, there will be one and only one such point of intersection, unless the lines are parallel, when there will be none. The equations (1) and (2) have, therefore, in general one and only one solution, or, in case the graphs are parallel lines, they have no solution.

Example 1. Solve graphically the simultaneous equations

$$\begin{cases} 2x - 3y = 7, & (1) \\ 3x + y = 5. & (2) \end{cases}$$

Solution. To make the graphs, we may determine the X- and Y-intercepts of each of the lines (1) and (2)[1]; in (1), when $y = 0$, $x = \frac{7}{2}$, and when $x = 0$, $y = -\frac{7}{3}$; in (2), when $y = 0$, $x = \frac{5}{3}$, and when $x = 0$, $y = 7$. As a check point take any value of x at random and compute the corresponding value of y; the point thus determined should lie upon the graph. If correctly drawn,

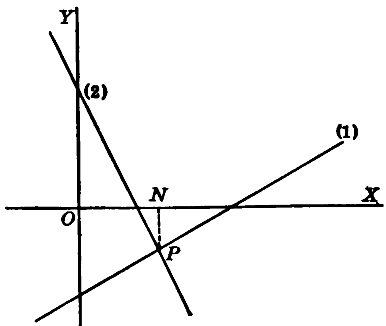


FIG. 21

[1] The expression "line (1)" is used for brevity, instead of the complete expression "line whose equation is (1)."

the lines (1) and (2) will result as in Fig. 21. The coördinates of the point of intersection P are evidently about $(2, -1)$, which fact gives (approximately) the solution of the problem, namely, $x = 2$, $y = -1$. In this case the graphical method has given us the *exact* solution, as we can easily verify by substituting $x = 2$, $y = -1$ in the equations (1) and (2).

Example 2. Solve graphically the simultaneous equations

$$\begin{cases} x + 4y = 10, & (1) \\ 5x - 8y = 1. & (2) \end{cases}$$

Solution. Making tables of values for x and y in order to determine points on the graph, we have in tabulated form:

(1)

			Check.
x	0	10	2
y	$2\frac{1}{2}$	0	2

(2)

			Check.
x	0	$\frac{1}{5}$	2
y	$-\frac{1}{8}$	0	$\frac{9}{8}$

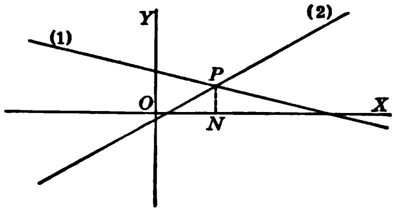


FIG. 22

The coördinates of the point of intersection P are about $(3, 2)$, which is in fact very nearly the correct solution, although not exactly so (as was the case in Ex. 1). Algebraic solution[1] shows that $x = 3$, $y = 1\frac{3}{4}$ is the exact solution.

Example 3. Solve graphically the simultaneous equations

$$\begin{cases} 3x - 4y = 5, & (1) \\ 6x - 8y = 7. & (2) \end{cases}$$

The graphs result as in Fig. 23, and apparently the lines are parallel. That they actually are so is shown by attempting an algebraic solution. If we multiply equation (1) by 2, we have

$$6x - 8y = 10,$$

whereas (2) requires that

$$6x - 8y = 7.$$

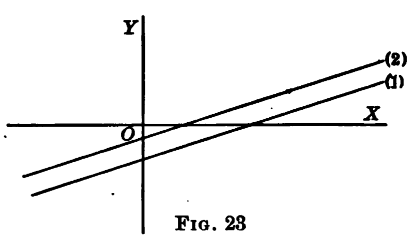


FIG. 23

Both cannot be true at once; hence there is *no* pair of values (x, y) that will satisfy *both* equations; that is, the lines (1) and (2) do not intersect, as was indicated by the figure.

[1] See Appendix E for the method, in case it has been forgotten.

EXERCISES

Solve the following pairs of simultaneous equations both graphically and algebraically:

1. $\begin{cases} 3x + 5y = 11, \\ x + y = 3. \end{cases}$

2. $\begin{cases} x - 4y = 1, \\ 2x + y = -7. \end{cases}$

3. $\begin{cases} x + 2y = 1, \\ 2x + 8y = 5. \end{cases}$

4. $\begin{cases} 4x - 7y = 11, \\ 3x + 2y = 1. \end{cases}$

5. $\begin{cases} 3x - y = 3, \\ 6x - 3y = 2. \end{cases}$

6. $\begin{cases} 15x + 7y = 11, \\ \frac{3}{7}x = 2y. \end{cases}$

7. $\begin{cases} 8x + 9y = 93, \\ 7x - 5y = 17. \end{cases}$

8. $\begin{cases} \frac{4}{5}x - \frac{2}{3}y = 7, \\ 6x - 5y = 20. \end{cases}$

9. $\begin{cases} 7.2x + 1.5y = 0.42, \\ 4.8x - 2.5y = 0.98. \end{cases}$

10. $\begin{cases} \dfrac{3x}{8} + \dfrac{4y}{5} = 18, \\ \dfrac{x}{4} + \dfrac{2y}{3} = 14. \end{cases}$

11. $\begin{cases} \dfrac{x-1}{5} - \dfrac{y+2}{6} = \dfrac{11}{30}, \\ \dfrac{2x+3}{7} - \dfrac{3y-1}{5} = 3. \end{cases}$

12. $\begin{cases} \dfrac{x}{3} + \dfrac{y}{4} = \dfrac{1}{4}, \\ \dfrac{4x}{3} - \dfrac{2y}{5} = \dfrac{3x}{4} + \dfrac{19y}{40}. \end{cases}$

13. $\begin{cases} \dfrac{x+3y}{8} - \dfrac{2x+y}{5} = \dfrac{x+y}{2} - \dfrac{3x-5y}{7}, \\ \dfrac{3x+2y}{7} + \dfrac{1-3y}{4} + \dfrac{3}{4} = \dfrac{5x+7}{5} + \dfrac{5x+8y}{2} + 4\frac{7}{20}. \end{cases}$

14. Prove algebraically that the simultaneous equations $ax + by + c = 0$ and $a'x + b'y + c' = 0$ will have one and only one solution unless $ab' = ba'$.

Determinants

28. Before passing on to another application of the graphical representation to elementary algebra, it will be useful to take up a new method of solving simultaneous linear equations. This new method is indeed a very valuable aid in many more advanced mathematical studies. It is the method of *determinants*, which are defined as follows:

A determinant is a symbolic expression in the form

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix},$$

which is understood to represent the quantity $a \cdot d - b \cdot c$. Thus,

$\begin{vmatrix} 3 & 4 \\ 2 & 5 \end{vmatrix}$ is a determinant and equals $3 \cdot 5 - 4 \cdot 2 = 15 - 8 = 7$; $\begin{vmatrix} x & y \\ y & x \end{vmatrix}$ is a determinant and represents $x^2 - y^2$. Before reading farther the student should write down some other determinants and give their values, so that this symbolic expression may become somewhat familiar.

29. Solution of simultaneous linear equations by use of determinants. If we solve by the ordinary elementary algebraic method the pair of equations

$$\begin{cases} ax + by = c, & (1) \\ a'x + b'y = c', & (2) \end{cases}$$

we obtain the values

$$x = \frac{cb' - bc'}{ab' - ba'}, \qquad y = \frac{ac' - ca'}{ab' - ba'}.$$

Now both numerator and denominator of each of these fractions are differences of products, and hence can be written in the determinant notation, as follows:

$$x = \frac{\begin{vmatrix} c & b \\ c' & b' \end{vmatrix}}{\begin{vmatrix} a & b \\ a' & b' \end{vmatrix}}, \qquad y = \frac{\begin{vmatrix} a & c \\ a' & c' \end{vmatrix}}{\begin{vmatrix} a & b \\ a' & b' \end{vmatrix}}.$$

These fractions can be written down by inspection of the equations (1) and (2) if we observe the following directions: The denominator is the same for x and y, and is formed by copying the *coefficients* of x and y in the equations, in the exact order in which they occur, $\begin{vmatrix} a & b \\ a' & b' \end{vmatrix}$. The numerator for x is formed by writing, *instead of the coefficients of* x, the numbers on the right-hand side of the equations (c and c'), thus: $\begin{vmatrix} c & b \\ c' & b' \end{vmatrix}$, the second column of the determinant being the coefficients of y (b and b'). Finally, the numerator for the value of y is formed by writing in the first column the coefficients of x in the equations (a and a'), and in the second column, instead of the coefficients of y, the numbers on the right-hand side of the equations (c and c'), thus: $\begin{vmatrix} a & c \\ a' & c' \end{vmatrix}$.

Example 1. Solve in this way the equations

$$\begin{cases} 3x - 7y = 2, & (1) \\ 4x + 5y = 17. & (2) \end{cases}$$

By the preceding rule the values of x and y are written down directly, thus:

$$x = \frac{\begin{vmatrix} 2 & -7 \\ 17 & 5 \end{vmatrix}}{\begin{vmatrix} 3 & -7 \\ 4 & 5 \end{vmatrix}} = \frac{10-(-119)}{15-(-28)} = \frac{129}{43} = 3,$$

$$y = \frac{\begin{vmatrix} 3 & 2 \\ 4 & 17 \end{vmatrix}}{\begin{vmatrix} 3 & -7 \\ 4 & 5 \end{vmatrix}} = \frac{51-8}{43} = 1.$$

The solution is accordingly (3, 1), which pair, in fact, satisfies both equations.

Example 2. Solve by determinants:

$$\begin{cases} \dfrac{x}{3} - \dfrac{y}{4} = 1, & (1) \\ y - x + 2 = 0. & (2) \end{cases}$$

Here the second equation is not in the form $a'x + b'y = c'$, so we first write it in that form: $-x + y = -2$. The solution can now be written down:

$$x = \frac{\begin{vmatrix} 1 & -\frac{1}{4} \\ -2 & 1 \end{vmatrix}}{\begin{vmatrix} \frac{1}{3} & -\frac{1}{4} \\ -1 & 1 \end{vmatrix}} = \frac{1-\frac{1}{2}}{\frac{1}{3}-\frac{1}{4}} = 6, \qquad y = \frac{\begin{vmatrix} \frac{1}{3} & 1 \\ -1 & -2 \end{vmatrix}}{\begin{vmatrix} \frac{1}{3} & -\frac{1}{4} \\ -1 & 1 \end{vmatrix}} = \frac{\frac{1}{3}}{\frac{1}{12}} = 4.$$

Therefore (6, 4) is the solution, and this pair of values satisfies both equations.

EXERCISES

Solve by the method of determinants the problems in the exercises on page 32.

***30.**[1] **Determinants of the third order.** This method of determinants can be used to solve a system of three linear equations in three unknown quantities. The determinants thus far used

[1] Starred paragraphs may be omitted if desired, without interfering with the unity of the course.

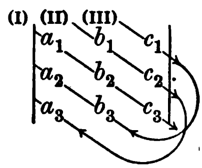
are called determinants of the second order, and we now define determinants of the *third* order as being symbolic expressions of the form

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}, \tag{1}$$

which is understood to represent the quantity

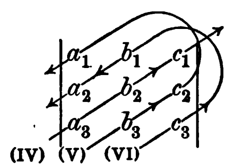
$$a_1 b_2 c_3 + b_1 c_2 a_3 + c_1 b_3 a_2 - a_3 b_2 c_1 - b_3 c_2 a_1 - c_3 b_1 a_2. \tag{2}$$

This rather long expression can be written down without burdening the memory at all, by observing that it contains three products with + sign and three products with − sign, and that the first three products can be read off by following the directions of the arrowheads in the following scheme:



(3)

From the line marked (I) we get the product $a_1 b_2 c_3$; from the line marked (II), the product $b_1 c_2 a_3$; and from the line marked (III), the product $c_1 b_3 a_2$. These are the three products that have the + sign in the value of the whole determinant. The other three products are read off in a similar way by following the direction of the arrowheads in this figure:



(4)

The lines marked (IV), (V), and (VI) give us the products $a_3 b_2 c_1$, $b_3 c_2 a_1$, and $c_3 b_1 a_2$, which, except that the negative sign must be prefixed to each, are the last three terms in the expression (2) for the whole determinant.

***31.** Two illustrative examples will make this procedure entirely clear, and will show that it is in reality very simple.

Example 1. Write the value of the determinant

$$\begin{vmatrix} 3 & 4 & 1 \\ 2 & 3 & 4 \\ 6 & 6 & 3 \end{vmatrix}.$$

It is better not to *draw* the guide lines marked (I), (II), (III), etc. in (3) and (4) above; we merely *think* them into the figure. Thus, the value of the determinant is

$$\begin{aligned} &3 \cdot 3 \cdot 3 + 4 \cdot 4 \cdot 6 + 1 \cdot 6 \cdot 2 - 6 \cdot 3 \cdot 1 - 6 \cdot 4 \cdot 3 - 3 \cdot 4 \cdot 2 \\ &= 27 + 96 + 12 - 18 - 72 - 24 = 21. \end{aligned}$$

Example 2. Write the value of the determinant

$$\begin{vmatrix} -1 & -6 & 4 \\ 2 & -3 & 5 \\ 1 & -2 & -1 \end{vmatrix}.$$

Here some of the numbers a, b, etc. are negative, but of course this fact introduces no new difficulty, except that *care* must be taken about the sign of each product. The result is

$$\begin{aligned} &(-1)(-3)(-1) + (-6) \cdot 5 \cdot 1 + 4(-2) \cdot 2 - 1(-3) \cdot 4 - (-2) \cdot 5(-1) \\ &-(-1)(-6) \cdot 2 = -3 - 30 - 16 + 12 - 10 - 12 = -59. \end{aligned}$$

EXERCISES

Write the value of each of the following seven determinants:

1. $\begin{vmatrix} 3 & 5 & 7 \\ 2 & 1 & 3 \\ 4 & 3 & 7 \end{vmatrix}.$

2. $\begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{vmatrix}.$

3. $\begin{vmatrix} 5 & 2 & -1 \\ -3 & -5 & -6 \\ -2 & 5 & 1 \end{vmatrix}.$

4. $\begin{vmatrix} 3 & -1 & 4 \\ 7 & -5 & -2 \\ 5 & -3 & 1 \end{vmatrix}.$

5. $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}.$

6. $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}.$

7. $\begin{vmatrix} x & y & z \\ 1 & 3 & -4 \\ 2 & -1 & 1 \end{vmatrix}.$

8. Prove that interchanging two adjacent rows (or two adjacent columns) of the terms of a determinant changes the sign of the determinant; that is,

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = - \begin{vmatrix} a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \\ a_2 & b_2 & c_2 \end{vmatrix} = + \begin{vmatrix} a_3 & b_3 & c_3 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} \text{ etc.}$$

(Note that the same statement can be made about a determinant of the second order as well.)

9. Prove that if any two rows (or two columns) in a determinant are identical, the determinant equals 0.

10. Prove that

$$\begin{vmatrix} a_1n & b_1n & c_1n \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = n \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix};$$

that is, if a number is a factor of every term in a single row (or column) of a determinant, it is a factor of the determinant.

11. Prove that

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 + na_2 & b_1 + nb_2 & c_1 + nc_2 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

12. A **minor determinant** is a determinant obtained from a given one by suppressing all the terms in any one single row and also all those in any one single column. Thus, $\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}$ is a minor determinant obtained from

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

by suppressing all the terms in the first row and all those in the first column. This minor determinant will be symbolized by A_1 and is said to "correspond" to the term a_1 at the *intersection* of the row and the column struck out. Similarly, to b_1 corresponds $B_1 = \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix}$ and so on. Prove that $a_1A_1 - b_1B_1 + c_1C_1 = a_1A_1 - a_2A_2 + a_3A_3 = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$, and also that $a_1B_1 - a_2B_2 + a_3B_3 = 0$.

***32. Solution of simultaneous linear equations in three variables.** If we solve, by the method of elementary algebra, the system of equations

$$\begin{cases} a_1x + b_1y + c_1z = d_1, & (1)\\ a_2x + b_2y + c_2z = d_2, & (2)\\ a_3x + b_3y + c_3z = d_3, & (3)\end{cases}$$

we obtain the values (after dividing out a common factor in numerator and denominator)

$$x = \frac{d_1b_2c_3 + b_1c_2d_3 + c_1b_3d_2 - d_3b_2c_1 - b_3c_2d_1 - c_3b_1d_2}{a_1b_2c_3 + b_1c_2a_3 + c_1b_3a_2 - a_3b_2c_1 - b_3c_2a_1 - c_3b_1a_2},$$

$$y = \frac{a_1d_2c_3 + d_1c_2a_3 + c_1d_3a_2 - a_3d_2c_1 - d_3c_2a_1 - c_3d_1a_2}{a_1b_2c_3 + b_1c_2a_3 + c_1b_3a_2 - a_3b_2c_1 - b_3c_2a_1 - c_3b_1a_2},$$

and a similar fraction for z.

Each numerator and denominator can be written in the form of a determinant, as follows:

$$x = \frac{\begin{vmatrix} d_1 & b_1 & c_1\\ d_2 & b_2 & c_2\\ d_3 & b_3 & c_3\end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1\\ a_2 & b_2 & c_2\\ a_3 & b_3 & c_3\end{vmatrix}},\quad y = \frac{\begin{vmatrix} a_1 & d_1 & c_1\\ a_2 & d_2 & c_2\\ a_3 & d_3 & c_3\end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1\\ a_2 & b_2 & c_2\\ a_3 & b_3 & c_3\end{vmatrix}},\quad z = \frac{\begin{vmatrix} a_1 & b_1 & d_1\\ a_2 & b_2 & d_2\\ a_3 & b_3 & d_3\end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1\\ a_2 & b_2 & c_2\\ a_3 & b_3 & c_3\end{vmatrix}},$$

where the denominator is in each case the determinant formed by copying the coefficients of x, y, and z in the equations (1), (2), and (3); and the numerators are formed in an exactly analogous way to that used in solving two equations in two variables.

Example. Solve the set of equations

$$\begin{cases} 2x - 3y + 5z = -10, & (1)\\ x + 2y - z = 9, & (2)\\ 5x - y + 3z = 7. & (3)\end{cases}$$

Writing down the values of x, y, and z according to the method just explained,

$$x = \frac{\begin{vmatrix} -10 & -3 & 5\\ 9 & 2 & -1\\ 7 & -1 & 3\end{vmatrix}}{\begin{vmatrix} 2 & -3 & 5\\ 1 & 2 & -1\\ 5 & -1 & 3\end{vmatrix}} = \frac{-60 + 21 - 45 - 70 + 10 + 81}{12 + 15 - 5 - 50 - 2 + 9} = \frac{-63}{-21} = 3,$$

$$y = \frac{\begin{vmatrix} 2 & -10 & 5 \\ 1 & 9 & -1 \\ 5 & 7 & 3 \end{vmatrix}}{-21} = \frac{54 + 50 + 35 - 225 + 14 + 30}{-21} = \frac{-42}{-21} = 2,$$

$$z = \frac{\begin{vmatrix} 2 & -3 & -10 \\ 1 & 2 & 9 \\ 5 & -1 & 7 \end{vmatrix}}{-21} = \frac{28 - 135 + 10 + 100 + 18 + 21}{-21} = \frac{42}{-21} = -2.$$

Therefore $x = 3$, $y = 2$, $z = -2$ is the solution of the problem, and in fact this set of values satisfies all three equations.

EXERCISES

Solve by the method of determinants:

1. $\begin{cases} x + 2y + z = 4, \\ 3x - y - 2z = -7, \\ x + y + 3z = 4. \end{cases}$

2. $\begin{cases} \dfrac{x}{5} + y + \dfrac{z}{2} = 18, \\ x + \dfrac{y}{4} + \dfrac{z}{3} = 20, \\ \dfrac{x}{3} + \dfrac{3y}{4} - z = 8. \end{cases}$

3. $\begin{cases} \frac{1}{2}x + \frac{1}{3}y + \frac{1}{4}z = 62, \\ \frac{1}{3}x + \frac{1}{4}y + \frac{1}{5}z = 47, \\ \frac{1}{4}x + \frac{1}{5}y + \frac{1}{6}z = 38. \end{cases}$

4. $\begin{cases} ax + by - cz = 2ab, \\ by + cz - ax = 2bc, \\ cz + ax - by = 2ac. \end{cases}$

5. $\begin{cases} (a + b)x + (a - b)z = 2bc, \\ (b + c)y + (b - c)x = 2ac, \\ (c + a)z + (c - a)y = 2ab. \end{cases}$

***33.** Linear equations in three variables do not lend themselves to graphical representation in the same way as linear equations in two variables, because a point is determined by a *pair* of values, — its abscissa and ordinate. In three-dimensional geometry, however, a set of *three* coördinates may be introduced, corresponding to the three variables x, y, and z, and then any equation containing these variables may be given a graphical interpretation. This is found very useful in all higher study of geometry, but we shall not make any use of it in this book.

THE QUADRATIC FUNCTION

34. The next application of the graphical representation to elementary algebra arises when we consider the *quadratic function.* We have drawn the graphs of a number of such functions (see

Exs. 14–23, p. 22), and in every case the locus was a curve of a particular form, which we called a *parabola.* We assume that this is *always* the case, just as we assumed that the graph of every linear function is a straight line, without being able as yet to prove the correctness of the assumption.[1] Stated in the form of a theorem the assumption is as follows:

The graph of every function of the second degree, that is, of the form $ax^2 + bx + c$ (or, what is the same thing, the graph of every equation of the form $y = ax^2 + bx + c$), **is a parabola.**

Knowledge of this fact will be useful in checking the accuracy with which the coördinates of points on the curve have been computed; for if the points, when plotted, do not lie on a curve of the general shape that we have learned to recognize as a parabola, a mistake has been made, either in the computing or in the plotting.

35. Intersection of parabola and straight line. Let us draw in the same figure the graphs of the following equations:

$$\begin{cases} y = x^2 - x - 4, & (1) \\ y = 2x. & (2) \end{cases}$$

The tables of values for determining points on the graphs are

(1)

x	$y = x^2 - x - 4$
0	-4
1	-4
2	-2
3	2
4	8
-1	-2
-2	2
$\frac{1}{2}$	$-4\frac{1}{4}$
$-\frac{1}{2}$	$-3\frac{1}{4}$

(2)

	x	$y = 2x$
	0	0
	1	2
Check.	2	4

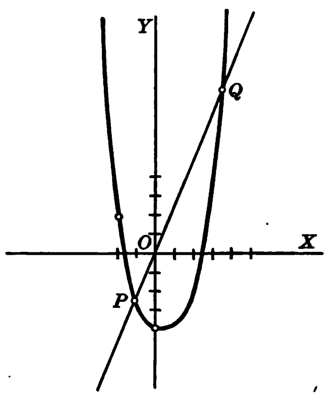


FIG. 24

Now the graph of (1) contains all points whose coördinates (x, y) satisfy that equation, and the graph of (2) contains all

[1] It is to be hoped that the student has not failed to notice that the word "parabola" has as yet no precise meaning at all; it is, so far, only a *word* used in order to give a name to a more or less vaguely defined curved line. For this reason, if for no other, it would be impossible at this point to *prove* the assumption above.

points whose coördinates satisfy the second equation. Hence any point whose coördinates satisfy *both* equations must be found on both graphs, that is, must be one of their points of intersection. By inspection of the figure we see that these points are approximately $(-1, -2)$ and $(4, 8)$; and in fact these are the *exact* points of intersection, for these pairs of values will satisfy both equations.

EXERCISES

Find in this way the points of intersection of the following:

1. $\begin{cases} y = x^2 + x - 3, \\ y = -x. \end{cases}$

2. $\begin{cases} y = 2x^2 - 5x + 1, \\ y = x - 3. \end{cases}$

3. $\begin{cases} y = 3x^2 + 2x - 1, \\ y = -3x + 1. \end{cases}$

4. $\begin{cases} y = 6x^2 - 8x - 12, \\ y = -x + 8. \end{cases}$

36. Algebraic solution. Let us return to the example above and consider how it could be solved *algebraically*. To this end we may substitute the value of y, obtained from one equation, in the other equation; thus, putting $2x$, the value of y obtained from the second equation, in place of y in the first equation, we have

$$2x = x^2 - x - 4;$$

that is, $$x^2 - 3x = 4.$$

This equation is an example of what is called in elementary algebra a "quadratic equation in one unknown quantity," and the student should be able to solve such an equation.[1] The result is $x = 4$, or $x = -1$. Since $y = 2x$, $y = 8$ when $x = 4$, and $y = -2$ when $x = -1$. Hence the solutions are $(4, 8)$ and $(-1, -2)$, exactly as we had already discovered by the graphic method.

EXERCISES

Solve the above four problems algebraically.

37. The general quadratic equation. Any quadratic equation in the one unknown quantity x can be reduced to the form

$$ax^2 + bx + c = 0, \tag{1}$$

[1] If not, sufficient time should be taken to review this process thoroughly; for this purpose Appendix F gives a full explanation, with examples.

which is accordingly a *general* quadratic equation. In this equation a, b, and c may be any numbers whatever, either rational or irrational (except that a cannot equal zero). Usually, however, these numbers will be rational, in the problems with which we shall have to deal. When we have solved the equation (1), we shall have a solution for every possible quadratic equation in one unknown quantity, since (1) represents *any* such equation. To solve (1), we use the method of elementary algebra known as "completing the square" (see Appendix F).

Dividing (1) by a and transposing, we have

$$x^2 + \frac{b}{a}x = -\frac{c}{a}. \tag{2}$$

To "complete the square," that is, to make the left side of (2) a perfect square, we must add $\left(\frac{b}{2a}\right)^2$:

$$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = \left(\frac{b}{2a}\right)^2 - \frac{c}{a};$$

that is,
$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a} = \frac{b^2 - 4ac}{4a^2}. \tag{3}$$

Therefore
$$x + \frac{b}{2a} = \pm\frac{\sqrt{b^2 - 4ac}}{2a},$$

or
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \tag{4}$$

These two values of x, then, $\frac{-b + \sqrt{b^2 - 4ac}}{2a}$ and $\frac{-b - \sqrt{b^2 - 4ac}}{2a}$, will satisfy the equation (1). The values of the unknown quantity which will satisfy a given equation are known as the *roots* of that equation. So the results (4) are the roots of the quadratic equation $ax^2 + bx + c = 0$. Since these are the roots of a *general* quadratic equation, the form (4) may be used as a symbolic representation, or *formula*, for the roots of *any particular* quadratic equation whatsoever, as the next paragraph illustrates.

38. Solution of quadratic equation by formula. If we take the equation that we have solved before (§ 36),

$$x^2 - 3x = 4,$$

we can write it in the form $ax^2 + bx + c = 0$ by mere transposition, thus:

$$x^2 - 3x - 4 = 0,$$

so that $a = 1$, $b = -3$, and $c = -4$. Now the roots of the equation $ax^2 + bx + c = 0$ are, as was just found,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a};$$

hence, using the values of a, b, and c that apply to our special equation,

$$x = \frac{-(-3) \pm \sqrt{(-3)^2 - 4 \cdot 1(-4)}}{2} = \frac{3 \pm \sqrt{9 + 16}}{2} = 4, \text{ or } -1,$$

which are of course the same results that we obtained before (§ 36).

Again, take the equation

$$x + \frac{1}{x} = 3.$$

Here the equation is easily reduced to the form $ax^2 + bx + c = 0$ by multiplying both sides by x and transposing, thus:

$$x^2 - 3x + 1 = 0.$$

By the formula the roots are

$$x = \frac{3 \pm \sqrt{9 - 4}}{2} = \frac{3 \pm \sqrt{5}}{2}.$$

(These results should be checked.)

EXERCISES

Solve by the formula the exercises on page 41, and also the following:

1. $3x^2 - 7x + 2 = 0.$
2. $7x^2 + 6x - 1 = 0.$
3. $4x^2 - x - 3 = 0.$
4. $10x^2 - 13x + \frac{5}{8} = 0.$
5. $x^2 - 6x - 775 = 0.$
6. $\dfrac{x+1}{x-1} + \dfrac{x+2}{x-2} = \dfrac{2x+13}{x+1}.$
7. $\dfrac{x+2}{x-1} - \dfrac{2-x}{x+1} = \dfrac{9}{4}.$

39. Solution by factoring. Suppose the quadratic function $ax^2 + bx + c$ can be resolved into two rational factors, thus:

$$ax^2 + bx + c = a(x - \alpha)(x - \beta). \qquad (1)$$

Now *a product can equal zero only if one of its factors equals zero*; hence, if $ax^2 + bx + c = 0$, that is, if $a(x - \alpha)(x - \beta) = 0$, then either

$$x - \alpha = 0$$

or

$$x - \beta = 0;$$

that is, $x = \alpha$ and $x = \beta$ are the only values of x that can satisfy the equation $ax^2 + bx + c = 0$. This means that α and β are the roots of the equation

$$ax^2 + bx + c = 0.$$

Example 1. Let us take once more the equation

$$x^2 - 3x - 4 = 0,$$

which we have already used several times. The factors of $x^2 - 3x - 4$ are

$$(x - 4)(x + 1).$$

Hence $x - 4 = 0$ and $x + 1 = 0$ are the only possible ways in which our equation can be satisfied. Thus the roots are 4 and -1, agreeing of course with the results already found.

Example 2. $5x^2 - 2x - 3 = 0.$

The factors of $5x^2 - 2x - 3$ are $(5x + 3)(x - 1)$; hence, if $5x^2 - 2x - 3 = 0$, then either $5x + 3 = 0$ or $x - 1 = 0$. Hence $x = -\frac{3}{5}$ or $x = 1$.

This method is evidently the simplest way that we have met of finding the roots of a quadratic equation. The only difficulty is that it is often impracticable to *find* the factors of the given quadratic function. In that case the previous method, that of the formula, should be used.

EXERCISES

Solve the following quadratic equations by factoring:

1. $x^2 + 3x + 2 = 0.$
2. $x^2 + \frac{1}{2}x - \frac{3}{2} = 0.$
3. $x^2 - 5x + 6 = 0.$
4. $x^2 - 5x - 6 = 0.$
5. $2x^2 - 5x + 2 = 0.$
6. $3x^2 - x - 2 = 0.$
7. $3x^2 + 2x - 1 = 0.$
8. $4x^2 - 20x + 9 = 0.$

40. Summarizing the results thus far obtained, we have seen that a quadratic equation in one unknown quantity can be solved (1) by "completing the square"; (2) by the formula; (3) by factoring. Of these only the third or the second is suited to practical use. Of course the unknown quantity need not be x, as it was in the examples we have considered; on the contrary, the student should practice solving quadratic equations in various letters, until the process becomes thoroughly familiar. A number of problems are added, as an aid in this work and for review purposes.

EXERCISES

Solve and check (algebraically):

1. $2t^2 + t - 2 = 0.$
2. $5y^2 - 3y - 2 = 0.$
3. $7t^2 - 8t + 8 = 0.$
4. $9y^2 - 11y - 12 = 0.$
5. $r + 3 = \dfrac{4}{r-1}.$
6. $s^2 - 10s + 16 = 0.$
7. $\dfrac{3}{r-6} - \dfrac{2}{r-5} = 1.$
8. $\dfrac{y}{100} + \dfrac{21}{25y} = -\dfrac{1}{4}.$
9. $\dfrac{1}{1-s^2} + \dfrac{1}{1-s} + \dfrac{1}{1+s} = 4.$

Solve the following pairs of simultaneous equations both algebraically and graphically:

10. $\begin{cases} y = x^2 - 5x, \\ y = 6. \end{cases}$
11. $\begin{cases} y = x^2 - 3x + 4, \\ y = 2. \end{cases}$
12. $\begin{cases} y = x^2, \\ y - x - 2 = 0. \end{cases}$
13. $\begin{cases} y = x^2, \\ x + y = 3. \end{cases}$
14. $\begin{cases} y = x^2 - 3x, \\ 2x + y = 2. \end{cases}$
15. $\begin{cases} y = 2x^2 - 6x + 1, \\ x + 2y = 3. \end{cases}$

41. Has every quadratic equation a solution? Let us try to solve this quadratic equation:

$$x^2 - 2x + 2 = 0. \tag{1}$$

Applying the formula, we get

$$x = \frac{2 \pm \sqrt{4-8}}{2} = \frac{2 \pm \sqrt{-4}}{2}. \tag{2}$$

Now (2) gives an *expression* for the roots of the equation (1); but what is the actual value of that expression? $\sqrt{4}$ is equal to 2 or to -2, since $2^2 = (-2)^2 = 4$; but what is the value of $\sqrt{-4}$? *Not* $-\sqrt{4}$, as is sometimes hastily concluded, for the simple reason that $-\sqrt{4} = -2$, and $(-2)^2$ is $+4$, not -4. As a matter of fact there *is no positive* and *no negative number* whose square is -4. Hence the "solution" (2) for x is merely a *formal expression,* without any value among all the numbers that we are acquainted with. The same thing is evidently true of *any* indicated square root of a negative number. We shall call such expressions "complex expressions." They are not, for us, subject to laws of operation, like numbers; but whenever they enter into a problem, we shall consider that the problem has *no solution.*

It is indeed found useful in higher mathematics to introduce *new numbers* such that their squares are negative; these new numbers are called "complex numbers," and simple, practical rules are adopted for performing arithmetical operations upon them. We shall not need to use them, however, and so we shall, as stated, consider as *unsolvable* any problem which leads to complex expressions.

We accordingly conclude that our quadratic equation

$$x^2 - 2x + 2 = 0$$

is unsolvable, because its roots are complex expressions.

EXERCISES

Show which of the following equations are unsolvable, that is, have complex roots:

1. $x^2 - x + 1 = 0$.
2. $2x^2 - 3x - 3 = 0$.
3. $2x^2 - 5x + 5 = 0$.
4. $x^2 + 3x + 2 = 0$.
5. $5x^2 + 3x - 1 = 0$.
6. $10x^2 - 7x + 2 = 0$.

42. Test for solvability of a quadratic equation. The situation described in the last paragraph can only happen when in the formula $\dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, for the roots of the equation, a

negative number appears under the square-root sign. But the quantity under the square-root sign is $b^2 - 4ac$; hence the roots will be complex when $b^2 - 4ac$ is *negative*. On the other hand, if $b^2 - 4ac$ is *positive*, the formula gives *two different values* for the roots of the equation, these values being *rational* if $b^2 - 4ac$ equals a perfect square, and *irrational* if $b^2 - 4ac$ is not a perfect square (on the assumption that a, b, and c are all rational numbers). One other possibility must not be overlooked, — that is, $b^2 - 4ac$ may *equal zero*; in this event the formula takes the very simple form $\dfrac{-b \pm \sqrt{0}}{2a}$, and thus there is *only one root*, $\dfrac{-b}{2a}$. We describe this situation by saying, "The roots of the equation $ax^2 + bx + c = 0$ are *equal* when $b^2 - 4ac$ equals zero."

Summarizing, we have the following facts:

$$\text{If } b^2 - 4ac \begin{cases} > 0, \text{ the equation has two unequal roots.} \\ = 0, \text{ the equation has equal roots.} \\ < 0, \text{ the equation is unsolvable (has } \textit{complex} \text{ roots).} \end{cases}$$

The quantity $b^2 - 4ac$ is called the **discriminant** of the quadratic equation $ax^2 + bx + c = 0$, since it enables us to determine the nature of the roots of the equation.

EXERCISES

Determine the nature of the roots of the following equations (without actually *finding* the roots):

1. $x^2 - 3x + 1 = 0.$
2. $x^2 - 4x + 4 = 0.$
3. $2x^2 - x - 1 = 0.$
4. $x^2 + x = 1.$
5. $3x^2 - 5x + 6 = 0.$
6. $4x^2 - 12x + 9 = 0.$
7. $5x^2 + 5x - 4 = 0.$
8. $\frac{1}{2}x^2 + 5x + 3 = 0.$
9. $8x^2 - 8x + 2 = 0.$
10. $\frac{2}{3}x^2 - \frac{3}{4}x + 1 = 0.$

43. Graphical interpretation of discriminant test. Let us return to the problems concerning the intersection of a parabola and a straight line, from which we were led to the study of quadratic equations. Take again the example on page 40, § 35.

Example 1.

$$\begin{cases} y = x^2 - x - 4, & (1) \\ y = 2x. & (2) \end{cases}$$

We saw that by eliminating y between the two equations there results the quadratic equation in x,

$$x^2 - 3x - 4 = 0, \tag{3}$$

which must be satisfied by the abscissas of the points of intersection of the graphs of (1) and (2). Now the discriminant of this equation is equal to $9 - 4(-4) = 25$, a positive number, and accordingly the equation has *two unequal roots*. Therefore there exist *two* abscissas (the roots of (3)) which determine points on *both* of the graphs; that is, the graphs intersect in two different points, as of course we proved by actually *finding* the points $(4, 8)$ and $(-1, -2)$. But the method of this section enables us to prove that the graphs really intersect, *without* actually finding the coördinates of their point of intersection, by merely noting the value of the discriminant b^2-4ac.

Example 2. Let us consider the graphs of the two equations

$$\begin{cases} y = 2x^2 - 3x - 4, & (1) \\ x - y - 6 = 0. & (2) \end{cases}$$

Substituting the value of y from (2) in (1), we have

$$2x^2 - 4x + 2 = 0,$$

that is, $x^2 - 2x + 1 = 0$, (3)

which must be satisfied by the abscissas of the points of intersection of (1) and (2). In (3), $b^2 - 4ac = 0$; hence there is *only one root* of (3), which means that there is *only one point* common to the parabola (1) and the straight line (2). In other words, the line is *tangent* to the parabola. The figure, of course, verifies this conclusion.

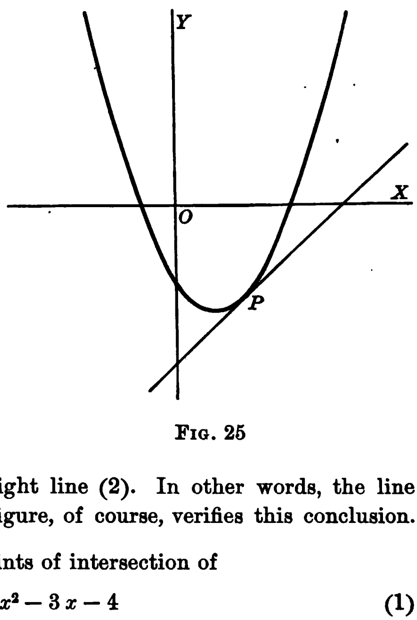


FIG. 25

Example 3. Let us find the points of intersection of

$$y = 2x^2 - 3x - 4 \tag{1}$$

and

$$y = 2x - 8. \tag{2}$$

Substituting the value of y from (2) in (1), we obtain

$$2x^2 - 5x + 4 = 0, \tag{3}$$

which must be satisfied by the abscissas of the points of intersection of (1) and (2). But in (3), $b^2 - 4ac = 25 - 32 = -7$; hence there is *no*

solution to (3), which means that there is *no point* common to the parabola (1) and the straight line (2). Moreover, a drawing of the two graphs verifies this conclusion, as of course it must. This verification is left to the student.

SUMMARY. Given two equations, representing a parabola and a straight line, after eliminating one of the variables from the two given equations we obtain a quadratic equation in one unknown quantity (the equation (3) in each of the above examples), whose roots are the abscissas (or ordinates) of the points common to the two graphs. If the discriminant of this equation is positive, the graphs intersect in two distinct points; if the discriminant equals zero, they have only one point in common, which means that the line is *tangent* to the parabola; and if the discriminant is negative, the line does not meet the parabola at all.

EXERCISES

Determine by the discriminant test, before drawing the graphs, whether each of the following pairs of lines will intersect, be tangent, or fail to meet. Verify by drawing the graphs.

1. $\begin{cases} y = 2x^2 - x - 3, \\ y = 3x. \end{cases}$

2. $\begin{cases} y = x^2 + x + 1, \\ y = 3. \end{cases}$

3. $\begin{cases} y = x^2 - 5x - 3, \\ y = -3\frac{1}{2}. \end{cases}$

4. $\begin{cases} y = x^2 - 4x + 2, \\ 2x + y + 1 = 0. \end{cases}$

5. $\begin{cases} y = x^2 + 3x + 1, \\ y = 4x - 3. \end{cases}$

6. $\begin{cases} y = x^2 - x + 1, \\ y = 0. \end{cases}$

7. $\begin{cases} y = x^2 - 2x - 3, \\ y + 4 = 0. \end{cases}$

8. $\begin{cases} y = 2x^2 - 3x + 8, \\ y = 5x. \end{cases}$

9. $\begin{cases} y = x^2 - 2x + 1, \\ y = 0. \end{cases}$

10. $\begin{cases} y = 3x^2 + 5x + 3, \\ x + y = 0. \end{cases}$

11. $\begin{cases} y = x^2 + x + 1, \\ 4y - 3 = 0. \end{cases}$

12. $\begin{cases} y = x^2 + 5x + 1, \\ y + 3 = 0. \end{cases}$

13. $\begin{cases} y = x^2 + x, \\ y - 2 = 0. \end{cases}$

14. $\begin{cases} y = x^2 + x, \\ y + 2 = 0. \end{cases}$

15. $\begin{cases} y = 3x^2 - 4x + 2, \\ x + y + 3 = 0. \end{cases}$

16. $\begin{cases} y = 4x^2 - 12x + 9, \\ y = 0. \end{cases}$

17. $\begin{cases} y = 4x^2 - 12x + 9, \\ y = 1. \end{cases}$

18. $\begin{cases} y = 4x^2 - 12x + 9, \\ y = -2. \end{cases}$

19. $\begin{cases} y = x^2, \\ y = 2x - 3. \end{cases}$

20. $\begin{cases} y = x^2, \\ x + y = 0. \end{cases}$

21. $\begin{cases} y = x^2 - 1, \\ x + y + 3 = 0. \end{cases}$

22. $\begin{cases} y = 2x^2 - 3x + 3, \\ x + y = 1. \end{cases}$

23. $\begin{cases} y = 2x^2 - 3x + 3, \\ x - y + 1 = 0. \end{cases}$

24. $\begin{cases} y = 5x^2 - 3x - 2, \\ y + x = 6. \end{cases}$

25. $\begin{cases} y = 4x^2 - 3x - 2, \\ y = x - 3. \end{cases}$

26. $\begin{cases} y = x^2 + x + 1, \\ y = \frac{1}{2}. \end{cases}$

44. Construction of a tangent to a parabola. This discriminant test is especially useful for *discovering tangents to a given parabola* (and also to certain other kinds of curves, as we shall see later). Thus, let us draw the parabola

$$y = x^2 + x + 2 \qquad (1)$$

and the straight lines

$$y = 3x + c \qquad (2)$$

for several values of c. Fig. 26 shows the lines for $c = 0$, $c = 1$, $c = 2$, and $c = -1$. We observe that the line for $c = 1$ seems to be tangent to the curve; to prove that it actually is so, we take equations (1) and (2) simultaneously, and, substituting $y = 3x + c$ in the equation $y = x^2 + x + 2$, we get

$$x^2 - 2x + 2 - c = 0, \qquad (3)$$

which has for its roots the abscissas of the points of intersection of (1) and (2). If (2) is *tangent* to (1), the roots of (3) will be *equal*, that is, $b^2 - 4ac$ will equal zero. But $b^2 - 4ac = (-2)^2 - 4(2 - c) = 4c - 4$, which will equal zero when $c = 1$. Hence the value $c = 1$ does in fact make the line (2) tangent to the parabola (1), just as was indicated by the graphical solution. We have thus found a line, namely, the line $y = 3x + 1$, which is tangent to the parabola $y = x^2 + x + 2$.

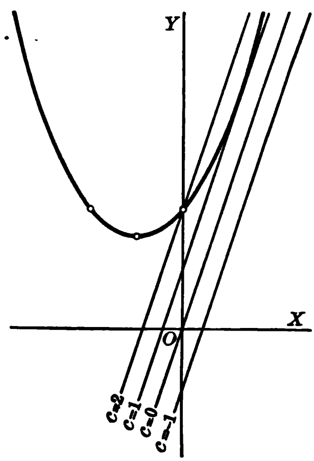


Fig. 26

Next, let us try the same parabola (1) with the lines

$$y = 2x + c. \tag{2'}$$

On drawing the lines (2′) for several values of c it will be seen that we have, as before, a set of parallel lines; but these make a smaller acute angle with the X-axis. Fig. 27 shows the lines for $c = 0$, $c = 1$, $c = 2$, and $c = 3$. The line for $c = 2$ seems to be tangent to the parabola, but we apply the discriminant test as above, to verify this conclusion. Eliminating y by substituting its value from (2′) in (1), we get

$$x^2 - x + 2 - c = 0. \tag{3'}$$

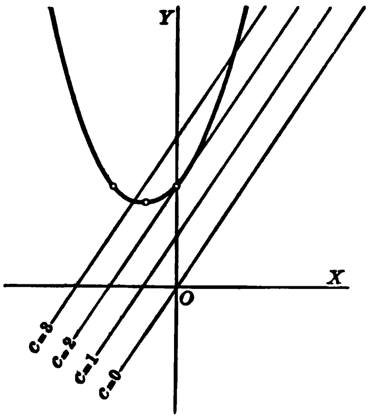


FIG. 27

The discriminant of (3′) is $4c - 7$, which will equal zero when and only when $c = \frac{7}{4}$. This shows that the line $y = 2x + \frac{7}{4}$ is tangent to the parabola, while the line $y = 2x + 2$ is not actually tangent. This illustrates again the fact that the graphical method cannot always be relied upon to give accurate results. Of course this is an unavoidable defect of graphic methods in general, whereas the algebraic method is *precise.*

EXERCISES

1. Find for what value of c the line $y = -x + c$ is tangent to the parabola $y = x^2 + x + 2$.

2. Answer the same question for the parabola $y = x^2 - 4x - 3$ and the line $y = \frac{1}{2}x + c$; for the line $y = -\frac{2}{3}x + c$; for the line $y = c$.

3. The same question for the parabola $y = 2x^2 - 3x + 2$ and the line $y = mx$. *Ans.* $m = 1$ or -7.

4. For what value of k will the line $y = mx + k$ (m being a *fixed* number) be tangent to the curve $y = x^2$?

5. Find the coördinates of the point where the tangent line of Ex. 4 meets the parabola. *Ans.* $\left(\frac{m}{2}, \frac{m^2}{4}\right)$.

6. Find the coördinates of the point where the tangent line of Ex. 4 meets the Y-axis. *Ans.* $\left(0, -\frac{m^2}{4}\right)$.

Draw a conclusion from the results to Exs. 5 and 6.

***45. Sets of curves.** In the equation $y = 3x + c$ of the example in § 44 the number c was understood to be capable of assuming any value we chose, thus giving rise to a set of straight lines. Such a general number as c in this equation, which can be given any particular value desired, is called a *parameter*. The presence of a parameter in an equation, then, indicates that the graphical representation of the equation will consist of an unlimited *set* of lines, straight or curved according to the nature of the equation. In § 44 we considered the problem of discovering which one of a set of straight lines is tangent to a given parabola, but the discriminant test can equally well be used to discover which one of a *set of parabolas* is tangent to a given straight line.

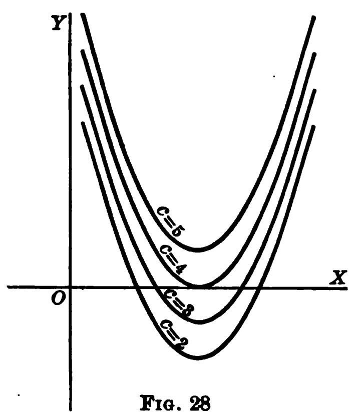


FIG. 28

Example 1. $\begin{cases} y = x^2 - 4x + c, & (1) \\ y = 0. & (2) \end{cases}$

Fig. 28 shows the parabolas for the values 2, 3, 4, and 5 of the parameter c. The curve for $c = 4$ seems to be tangent to the line (2), that is, to the X-axis; and the discriminant test verifies the result indicated by the figure. For, substituting in (1) the value $y = 0$ from (2), we get

$$x^2 - 4x + c = 0. \tag{3}$$

The discriminant of this is $16 - 4c$, which evidently equals 0 for $c = 4$, and only for that value.

Example 2.
$$\begin{cases} y = 2x^2 + kx + 8, & (1) \\ y = 0. & (2) \end{cases}$$

Putting $y = 0$ in (1), we have $2x^2 + kx + 8 = 0$. (3)

If (1) is tangent to (2), the discriminant of (3), which is $k^2 - 64$, must equal 0. This will happen when $k = \pm 8$, giving *two* curves of the set (1) that are tangent to (2), that is, to the X-axis. (If k is any number > 8, the discriminant of (3) will be *positive*, and the corresponding parabola will meet the X-axis in two distinct points; while if k is any number between -8 and $+8$, the discriminant will be *negative*, and hence the corresponding parabola will not meet the X-axis at all; if $k < -8$, there will again be two intersections.)

Draw the figure on a large scale.

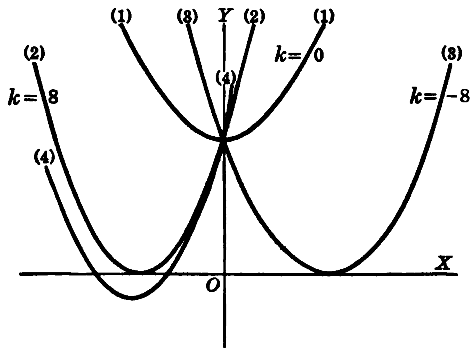


FIG. 29

EXERCISES

Find, by means of the discriminant test, for what values of the parameter k the curve and straight line will be tangent in each of the following exercises. Draw the graphs for these and also for several other values of k.

1. $\begin{cases} y = x^2 + 5x + k, \\ y = 0. \end{cases}$

2. $\begin{cases} y = x^2 - kx + 4, \\ y = 0. \end{cases}$

3. $\begin{cases} y = x^2 - kx + 12, \\ y = 3. \end{cases}$

4. $\begin{cases} y = x^2 - .1x + k, \\ y = 2. \end{cases}$

5. $\begin{cases} y = kx^2 - 3x - 5, \\ y = x + 1. \end{cases}$

6. $\begin{cases} y = kx^2 - x + 1, \\ y = 2x - 3. \end{cases}$

***46. Sum and product of roots of a quadratic equation.** The roots of the quadratic equation

$$ax^2 + bx + c = 0$$

are $\quad x_1 = \dfrac{-b + \sqrt{b^2 - 4ac}}{2a} \quad$ and $\quad x_2 = \dfrac{-b - \sqrt{b^2 - 4ac}}{2a}.$

The sum of the two is $\quad x_1 + x_2 = \dfrac{-2b}{2a} = \dfrac{-b}{a}, \qquad (1)$

and their product is

$$x_1 x_2 = \frac{b^2 - (b^2 - 4ac)}{4a^2} = \frac{4ac}{4a^2} = \frac{c}{a}. \qquad (2)$$

The results (1) and (2) enable us to write down *by inspection* the sum and the product of the roots of a quadratic equation.

Example 1. $2x^2 - 5x + 3 = 0.$

Here $\dfrac{-b}{a} = \dfrac{5}{2}$, which is accordingly the sum of the roots; and $\dfrac{c}{a} = \dfrac{3}{2}$, which is the product of the roots. We can verify this by actually finding the roots $2x^2 - 5x + 3 = (2x - 3)(x - 1) = 0$. Hence the roots are $x = 1$ and $x = \frac{3}{2}$. Their sum is in fact $\frac{5}{2}$, and their product $\frac{3}{2}$, as just found.

Example 2. Use the results (1) and (2) to find the values of $x_1^2 + x_2^2$ and of $\dfrac{1}{x_1^2} + \dfrac{1}{x_2^2}$ in terms of a, b, and c.

Solution. We have
$$x_1 + x_2 = \frac{-b}{a} \tag{1}$$
and
$$x_1 x_2 = \frac{c}{a}. \tag{2}$$

To get $x_1^2 + x_2^2$, begin by squaring (1):
$$x_1^2 + 2x_1x_2 + x_2^2 = \frac{b^2}{a^2}.$$

Multiply (2) by 2, $$2x_1x_2 = \frac{2c}{a}.$$

Subtracting, $$x_1^2 + x_2^2 = \frac{b^2 - 2ac}{a^2}. \tag{3}$$

To get $\dfrac{1}{x_1^2} + \dfrac{1}{x_2^2}$, which equals $\dfrac{x_1^2 + x_2^2}{x_1^2 x_2^2}$, we need only to divide (3) by $x_1^2 x_2^2$, that is, by $\dfrac{c^2}{a^2}$. This gives $$\frac{x_1^2 + x_2^2}{x_1^2 x_2^2} = \frac{b^2 - 2ac}{a^2} \div \frac{c^2}{a^2} = \frac{b^2 - 2ac}{c^2}. \tag{4}$$

EXERCISES

1. Write down the sum and the product of the roots of each of the following equations, and verify the results by actually *finding* the roots:

(1) $x^2 - 3x - 4 = 0.$
(2) $x^2 + 10x + 9 = 0.$
(3) $2x^2 - 3x - 2 = 0.$
(4) $5x^2 - 6x + 1 = 0.$
(5) $3x^2 - 11x + 6 = 0.$
(6) $2x^2 - 3x = 0.$
(7) $x^2 + px + q = 0.$
(8) $m^2x^2 + 2mx + 1 = 0.$

2. Write down the sum and the product of the roots of each of the following equations:

(1) $m^2x^2 + 2(m - 2a)x + 1 = 0.$
(2) $(b^2 + a^2m^2)x^2 + 2a^2mx + a^2(1 - b^2) = 0.$
(3) $(a + b + c)x^2 + (a + b - c)x + a^2 + b^2 - c^2 + 2ab = 0.$

3. Find two numbers whose sum is 22 and whose product is 117.

4. Find two numbers whose sum is 47 and whose product is 420.

5. Find the value of each of the following expressions in terms of a, b, and c (it being understood that x_1 and x_2 are the roots of the equation $ax^2 + bx + c = 0$):

(1) $x_1^3 + x_2^3$.
(2) $x_1^4 + x_2^4$.
(3) $x_1^5x_2^2 + x_1^2x_2^5$.
(4) $x_1^4x_2^7 + x_1^7x_2^4$.
(5) $\dfrac{1}{x_1^2x_2} + \dfrac{1}{x_1x_2^2}$.
(6) $\left(\dfrac{x_1}{x_2} + \dfrac{x_2}{x_1}\right)^2$.

***47. Factor theorem for quadratic equation.** On page 44 we learned that any *factorable* quadratic equation can be solved by inspection. We used the principle that "a product of two factors will equal zero when and only when one of the factors equals zero." The result of that investigation may be stated as follows:

If $x - \alpha$ is a factor of the function $ax^2 + bx + c$, then α is a root of the equation $ax^2 + bx + c = 0$. (I)

The object of this paragraph is to show that the *converse*[1] of this theorem is also true, that is, to prove:

If α is a root of the equation $ax^2 + bx + c = 0$, then $x - \alpha$ is a factor of the function $ax^2 + bx + c$. (II)

The statements (I) and (II) are called the *factor* theorem for the quadratic equation.

Proof of (II). $ax^2 + bx + c = a\left(x^2 + \dfrac{b}{a}x + \dfrac{c}{a}\right)$.

If α and β are the two roots[2] of the equation $ax^2 + bx + c = 0$, then, by the preceding paragraph,

$$\alpha + \beta = -\frac{b}{a}$$

and

$$\alpha\beta = \frac{c}{a}.$$

Therefore $ax^2 + bx + c = a[x^2 - (\alpha + \beta)x + \alpha\beta] = a(x - \alpha)(x - \beta)$. This proves the theorem.

[1] The *converse* of a theorem is a new theorem in which the hypothesis of the original theorem becomes the conclusion of the new one, and vice versa. Thus, the theorem "If the sides of a triangle are equal, its angles are equal" has for its converse the theorem "If the angles of a triangle are equal, its sides are equal." Here both the original theorem and its converse are true, but that is not always the case. The student should think of examples in which a theorem is true but its converse false.

[2] β will *equal* α in case $b^2 - 4ac = 0$, but the proof is valid in this case also. The case $b^2 - 4ac < 0$ cannot happen under the hypothesis we have made.

A second proof. Another method of proof of this theorem can be worked out by the student as follows: Divide $ax^2 + bx + c$ by $x - \alpha$, using the ordinary process of long division; the quotient will be found to be $ax + a\alpha + b$, with the remainder $a\alpha^2 + b\alpha + c$. Now, by the hypothesis, α is a root of the equation $ax^2 + bx + c = 0$; hence $a\alpha^2 + b\alpha + c = 0$; that is, the remainder when $ax^2 + bx + c$ is divided by $x - \alpha$ is 0. This proves the theorem.

A third proof. This theorem can also be proved in the following manner, which is especially instructive, and which, moreover, leads us to another result of importance.

Begin with $ax^2 + bx + c = a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right)$ as before. This time we will "complete the square" of the first two terms in the parenthesis, that is, add $\left(\frac{b}{2a}\right)^2$, so as to make a perfect square. We then have

$$\begin{aligned} a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right) &= a\left[x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} - \frac{b^2}{4a^2} + \frac{c}{a}\right] \\ &= a\left[\left(x + \frac{b}{2a}\right)^2 - \frac{b^2 - 4ac}{4a^2}\right] \qquad \text{(A)} \\ &= a\left[x + \frac{b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a}\right]\left[x + \frac{b}{2a} - \frac{\sqrt{b^2 - 4ac}}{2a}\right] \\ &= a\left[x - \frac{-b - \sqrt{b^2 - 4ac}}{2a}\right]\left[x - \frac{-b + \sqrt{b^2 - 4ac}}{2a}\right] \\ &= a(x - \alpha)(x - \beta), \end{aligned}$$

where $\alpha = \dfrac{-b - \sqrt{b^2 - 4ac}}{2a}$ and $\beta = \dfrac{-b + \sqrt{b^2 - 4ac}}{2a}$ are the roots of the equation $ax^2 + bx - c = 0$.

This completes the proof of the theorem. But it does more than that. The form (A) shows that the function $ax^2 + bx + c$ can be written as the *difference of two squares* whenever $b^2 - 4ac$ is *positive*; while if $b^2 - 4ac$ is *negative*, the form (A) will be the *sum* of two squares; and if $b^2 - 4ac$ equals zero, the function $ax^2 + bx + c$ is shown to be itself a perfect square. It follows from this that in each of the last two cases ($b^2 - 4ac \leqq 0$) the sign of the function $ax^2 + bx + c$ will be *the same as the sign of* a for all values of x (for neither the sum of two squares nor a single perfect square can be negative). This is often a useful fact to know concerning such a function. For example, the function $x^2 - x + 1$ will be *positive* for all values of x, since $b^2 - 4ac < 0$ and a is positive; and the function $-2x^2 + 10x - 13$ will be

negative for all values of x, since $b^2 - 4ac < 0$ and a is negative. In the case where $b^2-4ac>0$ the result is almost equally simple, for then (A) gives us $ax^2+bx+c=a(x-\alpha)(x-\beta)$, so that the sign of the function will be the *same* as that of a if both the factors $x-\alpha$ and $x-\beta$ are positive or if both are negative, that is, if $x>\beta$ or $<\alpha$; but the sign of the function will be *opposite* to that of a if $x-\beta$ is negative, while $x-\alpha$ is positive, that is, if x is *between* α and β ($\alpha<x<\beta$).

Example. $-x^2-5x-6=-1(x+3)(x+2)$.

Here $\alpha=-3$, $\beta=-2$, and evidently the sign of the function will be negative (the same as that of a) except when x is between -3 and -2 ($-3<x<-2$), in which case the last factor $x+2$ is negative; the factor $x+3$, however, is positive, so that their product is negative, and hence the function $-1(x+3)(x+2)$ is *positive*. This result is well illustrated by the graphic representation of the function. If we take as usual x for the abscissa and $y=f(x)=-x^2-5x-6$ for the ordinate, the graphical representation will give the parabola shown in Fig. 30. Of course $y=0$ when $x=-3$ or -2, that is, at the points A and B (because $x+3$ and $x+2$ are *factors* of $f(x)$); and we see that the function is *positive* (the curve is *above* the X-axis) for all values of x between -3 and -2, but that it is *negative* (the curve is *below* the X-axis) for all other values of x.

Table of values

x	y
0	-6
1	-12
-1	-2
-2	0
-3	0
-4	-2
$-2\frac{1}{2}$	$+\frac{1}{4}$
$-2\frac{1}{3}$	$+\frac{2}{9}$ etc.

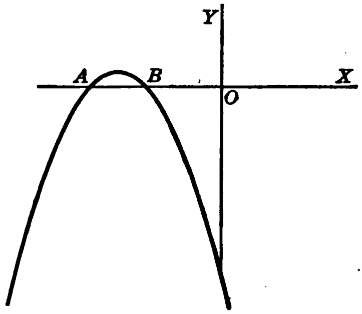


FIG. 30

EXERCISES

1. Test both algebraically and graphically the signs of the following functions:

(1) x^2-3x+3. (3) $x^2-10x-7$. (5) $3x^2-x+1$.
(2) x^2-3x+2. (4) $2x^2-3x-4$. (6) $5x^2-6x+1$.

2. Test algebraically in a similar manner the product

$$2(x-1)(x-2)(x-5),$$

and apply the information gained to make a rough sketch of the graph of the function, without computing any pairs of values exactly.

3. Test in the same way the product $2(x-1)^2(x-4)$; the product $-3(x+1)^2(x-3)^2$; the product $-x(x+1)(x+3)^2$.

***48. Maximum and minimum values of a quadratic function.** The expression (A) on page 56 enables us to draw still another conclusion. Since we have

$$ax^2+bx+c=a\left[\left(x+\frac{b}{2a}\right)^2-\frac{b^2-4ac}{4a^2}\right],$$

and since the term $\left(x+\frac{b}{2a}\right)^2$ is *always positive,* or *at least zero,* the least possible value of the whole expression in the square brackets will be obtained when $\left(x+\frac{b}{2a}\right)^2$ is *equal to zero*; that is, $x=-\frac{b}{2a}$ will give the *least possible value* of the expression which multiplies a, and therefore also the least possible value of ax^2+bx+c if a is positive. On the other hand, this value of x will obviously give the *greatest* possible value of ax^2+bx+c if a is negative, because the term $\left(x+\frac{b}{2a}\right)^2$ is then a *subtractive* term, so that *its* least value will give the *greatest* value of the whole function.

Example.
$$\begin{aligned}-2x^2+5x-1&=-2(x^2-\tfrac{5}{2}x+\tfrac{1}{2})\\&=-2[(x-\tfrac{5}{4})^2-\tfrac{25}{16}+\tfrac{1}{2}]\\&=-2[(x-\tfrac{5}{4})^2-\tfrac{17}{16}]\\&=-2(x-\tfrac{5}{4})^2+\tfrac{17}{8}.\end{aligned}$$

Since the *least* value that $(x-\frac{5}{4})^2$ can have is *zero*, $x=\frac{5}{4}$ will give the least value of that term, which, being subtractive, gives then the *greatest* value of the whole function. Thus, in this example the greatest value of the function is $\frac{17}{8}$, the value obtained when $x=\frac{5}{4}$. This agrees with the result just proved in general; namely, that $x=-\frac{b}{2a}$ will give the greatest value of the function, for here $\frac{-b}{2a}=\frac{-5}{-4}=\frac{5}{4}$.

It is always better to go through with the complete process of reducing the given function to the form (A) for each special case, as was done in the above example, rather than to use the value $x = \frac{-b}{2a}$ as a formula.

EXERCISES

Find the maximum or minimum values of each of the following functions, and verify your result by making a graph of the function:

(1) $x^2 - 5x + 6$. (3) $-x^2 - 6x + 7$. (5) $3x^2 - 6x + 5$.
(2) $2x^2 - 3x - 4$. (4) $-3x^2 + x - 1$. (6) $-x^2 - 10x + 1$.

NOTE. An easier method of finding maximum and minimum values of functions is developed in the Differential Calculus.

MISCELLANEOUS PROBLEMS INVOLVING QUADRATIC EQUATIONS

A few concrete problems leading to quadratic equations are here given. In solving such problems the most important thing is to be sure of understanding the problem itself thoroughly, then to translate the concrete language of the problem into the more abstract language of algebra.

1. The length of a rectangular field exceeds its breadth by 2 rd. If the length and the breadth were each increased by 4 rd., the area would be 80 sq. rd. Find the dimensions of the field.

2. The area of a certain square may be doubled by increasing its length by 10 ft. and its breadth by 3 ft. Find the length of its side.

3. A rectangular grass plot 12 yd. long and 9 yd. wide has a path of uniform width around it. The area of the path is $\frac{2}{3}$ of the area of the plot. Find the width of the path.

4. A farmer sold a number of sheep for $120. If he had sold 5 less for the same money, he would have received $2 more a head. How much did he receive per head?

5. A man agrees to do a piece of work for $48. It takes him 4 days longer than he planned, and he finds that he has earned $1 less per day than he expected. In how long a time did he plan to do it?

6. A circular grass plot is surrounded by a path of a uniform width of 3 ft. The area of the path is $\frac{7}{9}$ the area of the plot. Find the radius of the plot.

7. A boat steams down a river 12 mi. and back in 2 hr. 8 min., and its rate in still water is 4 times the rate of the current. Find the rate of the steamer and that of the current.

8. A straight line AB, of length l, is divided at a point X in such a way that AX is the mean proportional between AB and XB (that is, $AB : AX = AX : XB$). Find the lengths AX and XB. *Ans.* $AX = \dfrac{\sqrt{5}-1}{2}\,l.$

FIG. 31

AB is said to be *divided in extreme and mean ratio* at X. Prove that if BA is produced beyond A by a length $AX' = AX$, then $X'B$ is divided in extreme and mean ratio at A.

9. In a right triangle ABC whose hypotenuse AB equals 20, a perpendicular CD is drawn from C to AB. If $BC = \frac{2}{3}AD$, find the lengths of BC, AD, and AC.

10. Divide 100 into two such parts that their product shall be a maximum.

11. Find the dimensions of the largest rectangle that can be inclosed by a perimeter of 60 ft.

HINT. If x represents the number of feet in the length, then $30 - x$ will represent the number of feet in the breadth, and $x(30 - x)$ is to be made a maximum.

12. An isosceles triangle has the dimensions 10, 13, 13. Find the dimensions of the largest rectangle that can be inscribed in this triangle, one side of the rectangle lying in the base of the triangle.

13. A window is to be constructed in the shape of a rectangle surmounted by a semicircle. Find the dimensions that will admit the maximum amount of light, if its perimeter is to be 48 ft.

CHAPTER IV

INTRODUCTION TO THE TRIGONOMETRIC FUNCTIONS

49. Definition and measurement of angles. We have considered in some detail the question of measurement as applied to *distances*; we have now to consider its application to *angles*, — a question which is of equally fundamental importance. We shall find it convenient to think of an angle as resulting from a *rotation* of a straight line from one position to another, the vertex of the angle being the center of rotation. Thus, the angle BAC (Fig. 32) may be considered as generated by the rotation of a line about A, from the initial position AB to the terminal position AC; and the numerical measure of the angle BAC gives the *amount of this rotation* in terms of a conveniently chosen unit. The most common unit of measure is the *degree*, which is $\frac{1}{360}$ of a complete rotation about the vertex of the angle. Thus, an angle of 90°, or a *right angle*, is produced by one fourth of a complete revolution about the vertex; an angle of 180°, or a *straight angle*, is produced by one half of a complete revolution about the vertex; an angle of 60° is produced by one sixth of a complete revolution; and so on. Any angle can thus be measured by the amount of rotation from its initial line to its terminal line; and it is evident that an angle can have a numerical measure greater than 180° or even greater than 360°, for we can easily have an amount of rotation that is greater than 360°, that is, greater than one complete revolution. Thus, an angle of 450° would mean one complete revolution (360°) and 90° more.

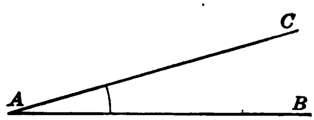


FIG. 32

50. Moreover, as rotation can take place in either of two opposite directions, we are led to make a distinction between *positive* and

negative angles. It is usual to regard angles that are produced by a *counterclockwise* direction of rotation as positive, and those that are produced by a *clockwise* direction of rotation as negative. Thus, in Fig. 33, if the rotation is *from AB to AC*, the angle is positive; if the rotation is *from AC to AB*, the angle is negative.

FIG. 33

EXERCISES

Construct (with a protractor) angles equal to 250°, – 30°, 390°, – 430°, – 400°, 1000°, – 180°, – 350°, 490°.

51. Relation to coördinate system. For convenience of reference the four parts into which the plane is divided by the X- and Y-axes are called *quadrants* and are numbered in the counterclockwise order, as shown in Fig. 34. If we now consider an angle to be formed by a rotation about the origin, the initial position being the positive half of the X-axis (OX), then the terminal lines of all angles between 0° and 90° will lie in the first quadrant; those of all angles between 90° and 180° will lie in the second quadrant; and so on. Thus, the terminal line of the angle 225° will be in the third quadrant, bisecting the angle between the X- and Y-axes (Fig. 35). We shall say, "The angle 225° is in the third quadrant," and similarly for any angle, meaning that the terminal line of the angle is in that quadrant when the initial line coincides with the positive half of the X-axis. Likewise, an angle of – 225° is in the second quadrant, and so on.

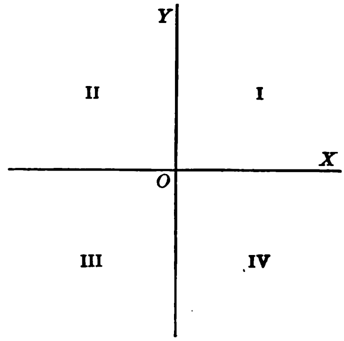


FIG. 34

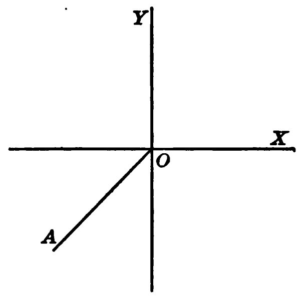


FIG. 35

EXERCISES

1. In what quadrant is each of the following angles: 150°, 300°, 550°, − 20°, − 150°, − 200°, 1040°, − 500°, − 36,280°? Using a protractor, construct each terminal line, and mention the number of complete revolutions in case it is more than one.

2. If a wheel makes 80 revolutions per minute, through how large an angle does it turn in 15 sec.? in $\frac{1}{3}$ sec.? in 5 min.? in 1 min. 50 sec.?

3. Through what angle does each hand of a clock rotate in 1 hr.? in 24 hr.?

52. Definition of the trigonometric functions. Having become familiar with the idea of an angle of any magnitude, including

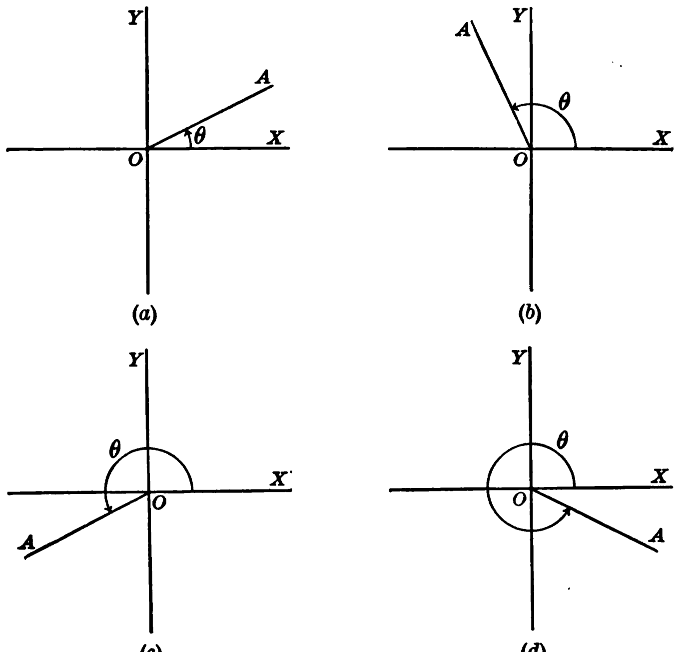


FIG. 36

the distinction between *positive* and *negative* angles, we are now prepared for the study of some important numerical values that depend upon angles, or are determined by angles.

Let θ[1] be any angle, and suppose it has been formed by a rotation from the initial position OX to the terminal position OA (Fig. 36). Let P be any point on the terminal line of the angle θ, and construct line segments representing the ordinate and the abscissa of P (Fig. 37). These segments are QP and OQ in each of the

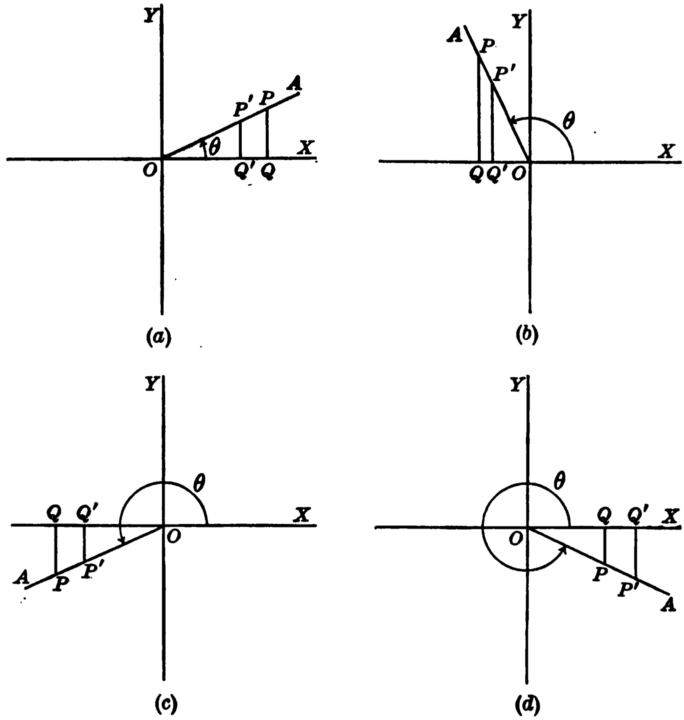


Fig. 37

four parts of Fig. 37. Now the *ratio* of QP to OQ will be the same for any one angle θ, *whatever point* P is taken on the terminal line OA. For if P' is any other point on OA, the triangle $OQ'P'$ is similar to the triangle OQP (why ?), and hence

$$\frac{Q'P'}{OQ'} = \frac{QP}{OQ}.$$

[1] See Greek alphabet at end of book.

In other words, the ratio $\frac{QP}{OQ}$, that is, the ratio $\frac{\text{ordinate of } P}{\text{abscissa of } P}$, depends for its value only upon the magnitude of the angle θ, and not upon the particular point P that we take. This ratio is accordingly a *function* of the angle θ. The same thing can evidently be said of the ratio $\frac{QP}{OP}$, which is the ratio $\frac{\text{ordinate of } P}{\text{distance from } O \text{ to } P}$, and also of the ratio $\frac{OQ}{OP}$, which is the ratio $\frac{\text{abscissa of } P}{\text{distance from } O \text{ to } P}$. (The distance from O to P is called the ***radius vector*** of P.)

All these ratios are thus *functions of the angle* θ. They are called the *trigonometric functions* and are named as follows (see

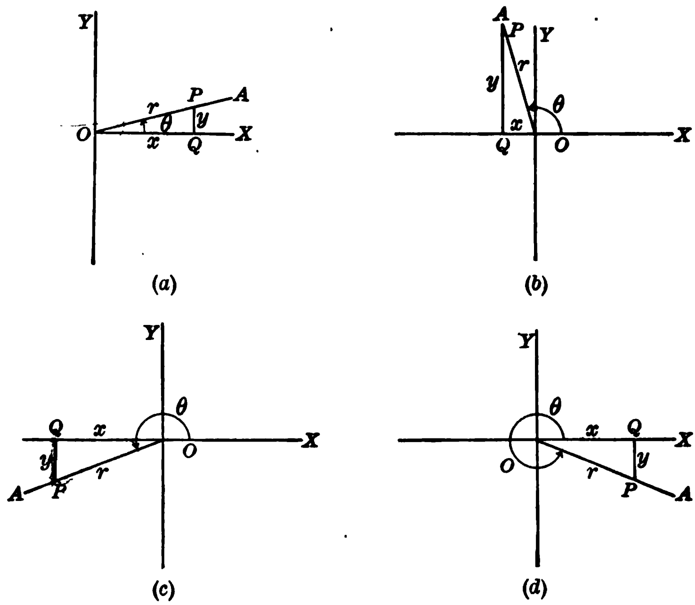


FIG. 38

Fig. 38, where x, y, and r symbolize the values of the abscissa, ordinate, and radius vector of the point P):

The ratio $\frac{\textbf{ordinate of } P}{\textbf{abscissa of } P} = \frac{QP}{OQ} = \frac{y}{x}$ is called the **tangent** of the angle θ and is written tan θ.

The ratio $\dfrac{\textbf{ordinate of } P}{\textbf{radius vector of } P} = \dfrac{QP}{OP} = \dfrac{y}{r}$ is called the **sine** of the angle θ and is written $\sin\theta$.

The ratio $\dfrac{\textbf{abscissa of } P}{\textbf{radius vector of } P} = \dfrac{OQ}{OP} = \dfrac{x}{r}$ is called the **cosine** of the angle θ and is written $\cos\theta$.

The reciprocals of these ratios are named as follows:

$\dfrac{1}{\tan\theta} = \dfrac{\textbf{abscissa of } P}{\textbf{ordinate of } P} = \dfrac{x}{y}$ is called the **cotangent** of the angle θ and is written $\cot\theta$.

$\dfrac{1}{\sin\theta} = \dfrac{\textbf{radius vector of } P}{\textbf{ordinate of } P} = \dfrac{r}{y}$ is called the **cosecant** of the angle θ and is written $\csc\theta$.

$\dfrac{1}{\cos\theta} = \dfrac{\textbf{radius vector of } P}{\textbf{abscissa of } P} = \dfrac{r}{x}$ is called the **secant** of the angle θ and is written $\sec\theta$.

These names must be thoroughly memorized and the definitions made familiar both in terms of the words used and in terms of the corresponding lines in the figure. Angles should be constructed in various positions, and the trigonometric functions of each one obtained. In doing this it must never be forgotten that both the ordinate and the abscissa of any point are *directed* distances, so that particular attention has always to be paid to the question of *sign*. It will be noticed that some of the ratios are negative on this account. The radius vector is always to be taken as *positive* in determining the signs of the ratios. For example, in Fig. 38, (*c*), $\cos\theta = \dfrac{x}{r}$, and x is negative; hence, r being positive, the ratio is negative, which means that the cosine of an angle in the third quadrant will be negative.

EXERCISES

1. Which of the ratios are negative when θ is in the second quadrant? when θ is in the third quadrant? in the fourth?

2. With the aid of a protractor, construct the following angles and determine as accurately as possible, by careful measurement, the values of the six trigonometric functions of each: 20°, 60°,

100°, 230°, 280°, 350°, $-30°$, $-300°$, $-1300°$, 2000°. State the results in *decimal* form.

3. Show that $\sin(-20°) = -\sin 20°$; $\sin(-70°) = -\sin 70°$; $\sin(-100°) = -\sin 100°$; $\sin(-210°) = -\sin 210°$. Generalize these results to apply to *any* angle α; that is, show that $\sin(-\alpha) = -\sin\alpha$.

4. Show that $\cos(-20°) = \cos 20°$; $\cos(-70°) = \cos 70°$; $\cos(-100°) = \cos 100°$; $\cos(-210°) = \cos 210°$; and generalize these results to apply to *any* angle α; that is, show that $\cos(-\alpha) = \cos\alpha$.

5. Prove that $\sin^2\alpha + \cos^2\alpha = 1$. (We usually write $\sin^2\alpha$, $\cos^2\alpha$, etc. for $(\sin\alpha)^2$, $(\cos\alpha)^2$, etc.)

Solution. $\sin\alpha = \dfrac{y}{r} = \dfrac{MP}{OP}$,

$$\cos\alpha = \frac{x}{r} = \frac{OM}{OP}.$$

Therefore

$$\sin^2\alpha = \frac{y^2}{r^2} \text{ and } \cos^2\alpha = \frac{x^2}{r^2};$$

hence $$\sin^2\alpha + \cos^2\alpha = \frac{y^2 + x^2}{r^2}.$$

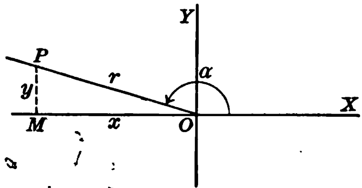


FIG. 39

But, by the Pythagorean Theorem, $y^2 + x^2 = r^2$.

Therefore $\sin^2\alpha + \cos^2\alpha = 1$. Q.E.D.

The student should draw figures with α in different quadrants, and show that the proof holds in these cases also.

6. Prove that $\tan^2\alpha + 1 = \sec^2\alpha$.

7. Prove that $\cot^2\alpha + 1 = \csc^2\alpha$.

8. Prove that $\dfrac{\sin\alpha}{\cos\alpha} = \tan\alpha$.

NOTE. The results of Exs. 5, 6, 7, and 8 are such important relations among the trigonometric functions that they should be memorized.

9. Construct an angle whose tangent is 2; find its value in degrees, with the aid of a protractor. What are the values of the other five trigonometric functions of this angle?

10. Construct an angle whose sine is $\frac{3}{5}$; find its value in degrees. Give the values of the other trigonometric functions of this angle.

11. Proceed as in Ex. 10, for an angle whose sine is $-\frac{3}{5}$; for an angle whose cosine is $-\frac{1}{2}$; for an angle whose tangent is $-\frac{4}{3}$.

53. Functions of 45°, 30°, and 60°. (I) If we construct an angle of 45°, we observe that the ordinate and the abscissa of any point on the terminal line are equal ($OQ = QP$, since the triangle OQP is isosceles); that is, $y = x$. Therefore

$$\tan 45° = 1 \quad \text{and also} \quad \cot 45° = 1.$$

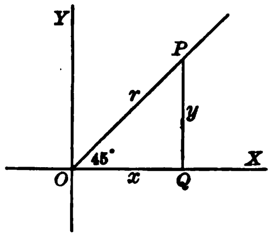


FIG. 40

Now $\quad r^2 = x^2 + y^2 = 2\,y^2;$

hence $\quad r = y \cdot \sqrt{2}.$

Therefore

$$\sin 45° = \frac{y}{y\sqrt{2}} = \frac{1}{\sqrt{2}} = \frac{1}{2}\sqrt{2},$$

and hence $\quad \csc 45° = \sqrt{2}.$

Since $\quad x = y,$

$$\cos 45° = \sin 45°.$$

Therefore $\quad \cos 45° = \dfrac{1}{\sqrt{2}} = \dfrac{\sqrt{2}}{2}.$

Hence $\quad \sec 45° = \sqrt{2}.$

(II) If we construct an angle of 30°, we find that $r = 2\,y$, as is easily seen by producing PQ its own length beyond OX (giving the point P') and joining OP', thus completing an equilateral triangle $OP'P$; whence

$$OP = P'P = 2\,y.$$

But $\quad x^2 = r^2 - y^2.$

Therefore $x^2 = 4\,y^2 - y^2 = 3\,y^2,$

and hence $\quad x = y \cdot \sqrt{3}.$

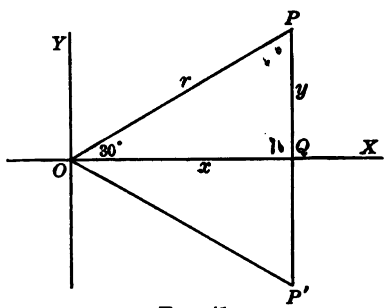


FIG. 41

Now the trigonometric functions of the angle 30° can be written down as follows:

$$\tan 30° = \frac{y}{x} = \frac{y}{y\sqrt{3}} = \frac{1}{\sqrt{3}} = \frac{1}{3}\sqrt{3},$$

$$\sin 30° = \frac{y}{r} = \frac{y}{2\,y} = \frac{1}{2},$$

$$\cos 30° = \frac{x}{r} = \frac{y\sqrt{3}}{2\,y} = \frac{\sqrt{3}}{2}.$$

The other three ratios can then be written down as reciprocals of these values respectively.

The values of the trigonometric functions of 60° can be obtained in a similar way. The work is left as a problem for the student.

54. Functions of $180° - \theta$ etc. (I) A very simple relation exists between the trigonometric functions of an angle θ and those of its supplement $180° - \theta$. Thus, let θ be any angle XOA', and let $\angle XOA = 180° - \theta$. Denoting BA by y, OB by x, and OA by r, as usual, we have, from the definitions of the trigonometric functions,

$$\tan(180° - \theta) = \frac{y}{x}. \quad (1)$$

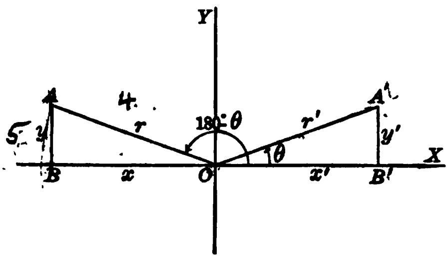


FIG. 42

Now take A' on the terminal line of θ so that $OA' = OA$, and draw $A'B'$ perpendicular to OX. Then the triangles OAB and $OA'B'$ are equal (being right triangles with the hypotenuse and an acute angle of one equal respectively to the hypotenuse and an acute angle of the other), so that $B'A' = BA$, that is, $y' = y$. Further, $BO = OB'$; and, since $x = OB$ (not BO), therefore $x = -x'$; and $r = r'$, since every radius vector is positive. Hence

$$\frac{y}{x} = \frac{y'}{-x'} = -\tan\theta.$$

Therefore, from (1),

$$\tan(180° - \theta) = -\tan\theta.$$

Again, $$\sin(180° - \theta) = \frac{y}{r} = \frac{y'}{r'} = \sin\theta;$$

that is, $$\sin(180° - \theta) = \sin\theta$$

and $$\cos(180° - \theta) = \frac{x}{r} = \frac{-x'}{r'} = -\cos\theta;$$

that is, $$\cos(180° - \theta) = -\cos\theta.$$

In the figure, θ was an angle in the first quadrant; but the proof holds, word for word, if θ is in any other quadrant. (If θ

is in the *third* quadrant, for example, $180° - \theta$ is constructed by rotating first through $180°$, then through an amount equal to $-\theta$, which will yield a terminal line in the *fourth* quadrant, as Fig. 43 shows. The student should draw figures illustrating the various possible values of θ and $180° - \theta$.)

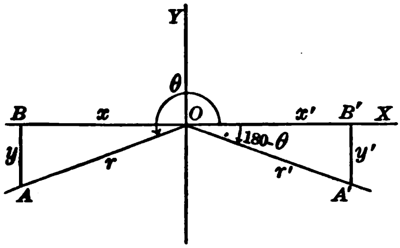


FIG. 43

(II) An equally simple relation exists between the trigonometric functions of an angle θ and those of the angle $90° + \theta$.

Let θ be any angle, and let $\angle XOA = 90° + \theta$. Then (Fig. 44)

$$\tan(90° + \theta) = \frac{y}{x} = \frac{BA}{OB}. \qquad (1)$$

Now take A' on the terminal line of θ so that $OA' = OA$, and draw $A'B'$ perpendicular to OX. Then the triangles OAB and $OA'B'$ are equal (why?), and hence $BA = OB'$, that is, $y = x'$; also $BO = B'A'$, that is, $x = -y'$ (for $BO = -x$ and $B'A' = y'$). Hence

$$\frac{y}{x} = \frac{x'}{-y'} = -\cot\theta. \qquad (2)$$

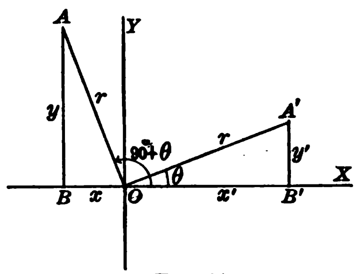


FIG. 44

Comparing (1) and (2),

$$\tan(90° + \theta) = -\cot\theta.$$

Similarly, $$\sin(90° + \theta) = \frac{y}{r} = \frac{x'}{r'} = \cos\theta$$

and $$\cos(90° + \theta) = \frac{x}{r} = \frac{-y'}{r'} = -\sin\theta.$$

Fig. 44 shows in this case also an angle θ in the first quadrant, but, as in (I), the student should construct figures showing that the proof still holds in case θ is in the second or any other quadrant.

EXERCISES

1. Prove that $\sin(180° + \theta) = -\sin\theta$, $\cos(180° + \theta) = -\cos\theta$, $\tan(180° + \theta) = \tan\theta$.

2. Prove that $\tan(270° - \theta) = \cot\theta$, $\sin(270° - \theta) = -\cos\theta$, $\cos(270° - \theta) = -\sin\theta$.

3. Give the exact values of the following: $\sin 120°$, $\tan 150°$, $\sin(-120°)$, $\tan(-150°)$, $\tan 225°$, $\cos 240°$, $\sin 300°$, $\tan 300°$, $\cos 330°$, $\cos(-300°)$.

4. Show that $\sin 150° + \cos 240° = 0$.

5. Show that $\tan 60° + \sin 240° + \cos 150° = 0$.

6. Show that $\sin 150° \cdot \cos 60° + \sin 60° \cdot \cos 150° = \sin 210°$.

7. Show that $\cos 330° \cdot \cos 210° + \sin 330° \cdot \sin 210° = \cos 120° = -\frac{1}{2}$.

8. Show that $\dfrac{\sin 120°}{1 + \cos 120°} = \tan 60°$.

9. Show that $2 \cdot \sin 120° \cdot \cos 120° - \sin 240° = 0$.

10. Show that $\sin 0° = 0$, $\cos 0° = 1$, $\tan 0° = 0$.

11. Show that $\sin 90° = 1$, $\cos 90° = 0$, $\cot 90° = 0$.

Applications of the Trigonometric Functions

55. Having learned the meaning of the trigonometric functions, we now take up the question of their applications. These are very numerous, large parts of physics, surveying, and astronomy being indeed based entirely on the use of these functions. We shall here consider only a few of the simplest applications.

56. Solution of the right triangle. To "solve" a triangle means to find the unknown parts (angles, sides, etc.) from sufficient data. But what are "sufficient data"? Elementary geometry answers this question for us by showing what combinations of sides and angles are sufficient to determine a triangle. In the case of the right triangle it is proved that any of the following combinations is sufficient:

1. A leg and an acute angle.
2. A leg and the hypotenuse.
3. The hypotenuse and an acute angle.
4. The two legs.

EXERCISES

1. Construct accurately, with rule and compass, a right triangle corresponding to each of the four problems indicated in § 57.

2. State what would be "sufficient data" for constructing an isosceles triangle; a scalene triangle not right-angled. Make constructions for each case.

57. Notation. The notation of Fig. 45 will be found convenient and will be used throughout this section. The angles α, β, and γ are opposite the sides a, b, and c respectively, and γ is the right angle.

58. The way in which right triangles may be solved by the help of the trigonometric functions may be best explained by giving some examples.

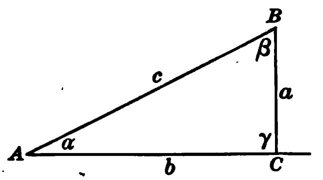


FIG. 45

Example 1. In the right triangle ABC, given $b = 150$, $\alpha = 75°$, to find the unknown parts.

Solution. Since the value of the angle α is given, all of its trigonometric functions are determined, and can be computed after the manner of Ex. 2, p. 66; or their values may be read out of a printed "table of trigonometric functions." From now on we shall use the latter method.[1] We find $\tan 75° = 3.7321$. But $\tan \alpha = \frac{a}{b}$, and hence we have

$$\tan 75° = 3.7321 = \frac{a}{150}. \tag{1}$$

Therefore $$a = 150 \cdot 3.7321 = \underline{\underline{559.8}}.$$

Next, to find the value of c:

$$\cos \alpha = \frac{b}{c} = \cos 75° = 0.2588. \tag{2}$$

Therefore $$\frac{150}{c} = 0.2588, \quad c = \frac{150}{.2588} = \underline{\underline{579.6}}.$$

Finally, as a check on the accuracy of these results,

$$\sin \alpha = \frac{a}{c} = \frac{559.8}{579.6} = .9658.$$

[1] Any one of the printed tables on the market may be used. Four decimal places give ample accuracy for this work.

Referring to the table, we find $\sin 75° = .9659$. Thus the results check. (A discrepancy of 1 or 2 in the fourth place may be expected when four-figure tables are used.)

Another method of checking would be to use the Pythagorean Theorem, $c^2 = a^2 + b^2$; this also verifies the correctness of the results obtained above.

59. We observe that the method of solving such problems consists in writing down the value of one of the trigonometric functions of the given acute angle, being careful to choose a ratio that contains the given side. Thus, in the above example we started with $\tan \alpha = \frac{a}{b}$ rather than with $\sin \alpha = \frac{a}{c}$, because neither a nor c was given. In case two sides are given, the modification of the method to be used is nearly self-evident and is illustrated by the following example:

Example 2. In a right triangle ABC, given $a = 15$, $c = 20$, to solve the triangle.

Since the two sides a and c are known, we write down a trigonometric function which contains both a and c:

$$\frac{a}{c} = \sin \alpha. \tag{1}$$

Therefore $$\sin \alpha = \tfrac{15}{20} = .75.$$

Using the tables, we find that the angle whose sine is .75 is $48° 35'$; hence

$$\alpha = \underline{\underline{48° 35'}}.$$

Therefore $$\beta = \underline{\underline{41° 25'}}.$$

To find b, $$\frac{b}{c} = \cos \alpha = \cos 48° 35' = .6615.$$

Therefore $$b = 20 \cdot .6615 = \underline{\underline{13.23}}.$$

Check. $$\tan \alpha = \frac{a}{b} = \frac{15}{13.23} = 1.134.$$

And as 1.134 is in fact the value of $\tan 48° 35'$, the above results are checked.

60. In dealing with right triangles in practice, it will not always be convenient to turn the figure about so that the vertex of the acute angle we are using shall fall on the origin, and one side

along the X-axis. Thus, in Fig. 46, to write down the ratios that form the trigonometric functions of the angle α we should consider A to be the origin and AC to be the X-axis, when of course AC would be the abscissa, and CB the ordinate, of the point B. Then

$$\tan\alpha = \frac{CB}{AC} = \frac{a}{b}, \quad \sin\alpha = \frac{CB}{AB} = \frac{a}{c}, \text{ etc.}$$

This is not altogether an easy process, however, especially for such an angle as β in Fig. 46, and we shall find it more practical to think of the definitions of the trigonometric functions *in terms of the sides of the triangle themselves.* We shall then regard a as "the side opposite the acute angle α," b as "the side adjacent to the angle α," and c as "the hypotenuse of the right triangle." Using these terms, we can easily restate the definitions of the trigonometric functions for an acute angle in a right triangle in such a way that we shall not need to think of an X- or Y-axis at all. Moreover, since the angles concerned are necessarily acute angles, we shall have *positive* values for all the trigonometric functions, and so may think of each side as undirected; that is, we may take its length as positive. Careful consideration will show that, in whatever position the angle may appear, the values of the trigonometric functions may be stated as follows:

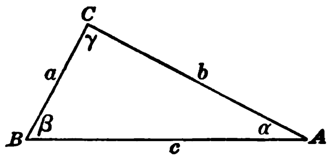


FIG. 46

$$\tan\alpha = \frac{\text{side opposite } \alpha}{\text{side adjacent } \alpha} = \frac{a}{b},$$

$$\sin\alpha = \frac{\text{side opposite } \alpha}{\text{hypotenuse}} = \frac{a}{c},$$

$$\cos\alpha = \frac{\text{side adjacent } \alpha}{\text{hypotenuse}} = \frac{b}{c}.$$

And the values of $\cot\alpha$, $\csc\alpha$, and $\sec\alpha$ are the reciprocals of these three respectively. This way of thinking of the trigonometric functions will be found useful in problems that involve right triangles, but it must not be forgotten that it applies *only* to right triangles.

EXERCISES

1. Construct a right triangle with the sides 3, 4, and 5, and give the values of the six trigonometric functions of each acute angle.

2. Solve the same problem, the given sides being 5, 12, and 13; 8, 15, and 17.

3. In Fig. 46, read off the values of the functions of α and of β.

4. Compare the values of the functions of α with those of β, and thus show that for any acute angle α the following relations hold:

$$\sin(90° - \alpha) = \cos\alpha,$$
$$\cos(90° - \alpha) = \sin\alpha,$$
$$\tan(90° - \alpha) = \cot\alpha.$$

5. Prove that the results of Ex. 4 are true for *any* angle α.

6. In Fig. 47, $\angle ACB = 90°$ and $\angle ADC = 90°$; show that the three right triangles formed are similar, and hence write down the trigonometric functions of the angles α and β, each in three ways.

Thus, $\sin\alpha = \dfrac{h}{b} = \dfrac{q}{a} = \dfrac{a}{p+q}$.

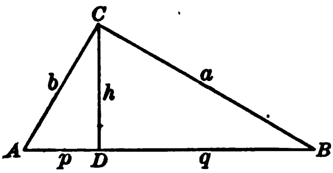


FIG. 47

In each of the following problems, through Ex. 22, construct the triangle determined by the given data, and make an estimate of the values of the unknown parts; then compute these values and check the results arithmetically.

7. $a = 6$ in., $\alpha = 30°$.

8. $b = 75$ ft., $\beta = 15°$.

9. $c = 1.3$ ft., $a = .9$ ft.

10. $a = 1$ ft., $b = 2$ ft.

11. $b = 2\frac{3}{4}$ in., $c = 5$ in.

12. $\alpha = 67°$, $a = 356$ ft.

13. $\beta = 88°$, $c = 110$ ft.

14. $a = 15.8$ mi., $b = 6.3$ mi.

15. $a = 72$ ft., $\beta = 25°$.

16. $c = 10$ in., $\alpha = 70°$.

17. $c = 40$ in., $a = 6$ in.

18. $a = 1$ ft., $c = 2$ ft.

19. $\alpha = 40°$, $a = 1$ mi.

20. $\beta = 16°$, $b = 23.4$ ft.

21. $\alpha = 73°$, $b = 17.3$ ft.

22. $\alpha = 42°$, $c = 3950$ mi.

Each of the following problems depends for its solution upon the solution of a right triangle. In most cases merely drawing the figure will give the clue to the method to be employed. One technical term needs explanation; the "angle of elevation" of an object means the angle formed by the line of sight to the object, and the horizontal line, — the angle CBA in Fig. 48.

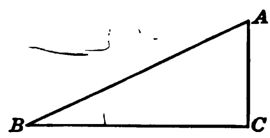


Fig. 48

23. Determine the height of a tree if its angle of elevation, from a point 200 ft. away, is 20°.

24. A standpipe 100 ft. high stands on the bank of a river. From a point directly opposite, on the other bank, the angle of elevation of the standpipe is 29°. How wide is the river there?

25. A rectangle is 40 in. $\times$ 17 in. What angle does the diagonal make with the longer side?

26. In a circle of radius 5 in., how long is a chord that subtends an angle of 20° at the center?

27. How long is the chord that subtends an angle of 1° at the center of a circle of radius 100 ft.?

28. Find the side and the area of a regular nonagon (9-sided polygon) inscribed in a circle of radius 16 ft.; circumscribed about the same circle.

29. Solve the same problem for the regular dodecagon (12-sided polygon).

30. Solve the same problem for the regular 15-sided polygon.

31. Find the radius of the fortieth parallel of latitude; of the eighty-fifth. (Assume the earth a sphere of radius 3950 mi.)

32. Find the length of the perimeter of a regular inscribed polygon of 24 sides when the diameter of the circle equals 1. Solve the same problem for the regular circumscribed polygon of 24 sides.

33. Solve the problem of Ex. 32 for the regular inscribed and circumscribed polygons of 48 sides.

34. Solve the problems of Exs. 32, 33 for the regular inscribed and circumscribed polygons of 96 sides (given $\sin 1\frac{7}{8}° = .032719$ and $\tan 1\frac{7}{8}° = .032737$).

35. Use the results of Ex. 34 to determine an approximate value of π. *Ans.* π is between 3.1410 and 3.1428.

61. Velocities and forces. A second useful application of the trigonometric functions is to physical problems involving velocities or forces. These are indeed, as we shall see, merely special cases of the solution of right triangles, but as they involve certain difficulties of their own, it is better to consider them in a separate section.

62. Suppose a body is moving with known velocity v in the direction AB, making an angle θ with the northerly direction, as in Fig. 49. Then the problem arises, How fast is the body moving toward the east? If AB represents the distance traveled in a *unit* of time, then $AB = v$. If AN represents the direction *north* from A, the angle NAB will equal the given angle θ. Drawing BC perpendicular to AN, we have a right triangle ABC in which $CB = x$ represents the distance the body has traveled *toward the east* in unit time; that is, x is the numerical value of the required velocity toward the east. Hence the solution of the problem is the same thing as the solution of the right triangle ABC. The side $AC = y$ will give the numerical value of the velocity toward the north. x and y are called the eastern and northern **components** of the given velocity v; v is called the **resultant** of the velocities x and y. Note that $x^2 + y^2 = v^2$.

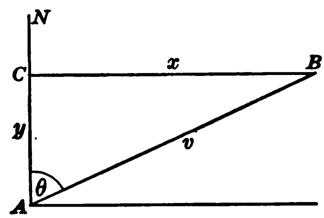


FIG. 49

Directions are usually given with reference to north or south as standards; thus, the direction 20° east of north is written N. 20° E. (read "North, 20° east"), and the direction 20° south of east, S. 70° E. (not E. 20° S.).

EXERCISES

1. A train is running at a speed of 40 mi. per hour in a direction 10° north of east (N. 80° E.). How fast is it moving *eastward*, and how fast *northward*?

2. A man walks at the rate of $3\frac{1}{2}$ mi. per hour in the direction S. 36° E. How far south has he gone in 3 hr.?

3. A boat is steaming in a direction N. 70° E. at the rate of 20 knots per hour. What are the eastern and northern components of this velocity?

4. A point moves in a vertical plane at the rate of 20 ft. per second in a direction inclined 53° with the horizontal. Find the horizontal and vertical components of this velocity.

5. The horizontal and vertical components of a velocity are respectively 30 ft. per second and 40 ft. per second. Find the resultant velocity and its direction.

6. A balloon is rising vertically at the rate of 660 ft. per minute and encounters a wind blowing horizontally at the rate of 15 mi. per hour. In what direction will the balloon continue to rise and with what velocity?

7. From the platform of a train going at the rate of 40 mi. per hour a boy throws a stone in the direction at right angles to the train's motion with a velocity of 50 ft. per second. In what direction will the stone go, and how fast?

8. A projectile from an 8-in. gun on a warship has a velocity of 2000 ft. per second, and the ship is moving 22 knots per hour (1 knot = 6080 ft.). If the gun is fired in a direction perpendicular to the ship's motion, in what direction will the projectile actually go?

9. A man rows at the rate of 4 mi. per hour, and the rate of the current in a river is 3 mi. per hour. If he starts to row straight across at a point where the river is 350 ft. wide, how far down will he reach the other bank?

10. If in Ex. 9 the man wishes to land directly opposite his starting-point, in what direction must he row?

11. A hunter is traveling straight north in an auto at the rate of 20 mi. per hour, when he notices a rabbit in a field about 100 ft. away. He fires when he is due east of the rabbit. If the velocity of the shot is 1000 ft. per second, in what direction must he aim if he is not to miss?

63. Problems involving component *forces* are identical mathematically with these problems in velocities. For example, if two forces, one of 40 lb. and the other of 30 lb., act at right angles to each other upon a point P, the effect is equivalent to that of a single force F acting upon the point P in the direction PQ, the diagonal of the rectangle $PAQB$, and with an intensity

numerically equal to the length of PQ. F is called the *resultant* of the component forces PA and PB. Here, evidently, $F = 50$ lb., and by solving the right triangle PAQ we find $\theta = 36° 52'$.

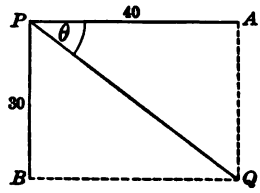


FIG. 50

This relation between the component and resultant forces is familiar to students of physics under the name "parallelogram of forces." It may be stated as follows: If a point P is acted upon by two forces, represented in magnitude and direction by PA and PB, then the diagonal PQ of the parallelogram $PAQB$ represents, in magnitude and direction, the resultant force.

EXERCISES

1. Two forces, of 45 lb. and 75 lb., act at right angles to each other upon a point. Find the direction and intensity of the resultant force.

2. The resultant of two forces at right angles to each other is 100 lb., and it makes an angle of 30° with the horizontal force. Find the horizontal and vertical components.

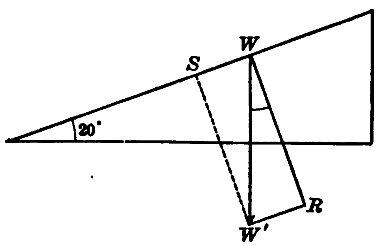


FIG. 51

3. A weight of 250 lb. lies on an inclined plane whose angle is 20°. With what force does it press against the side of the plane, and with what force does it tend to slide down the plane? (In Fig. 51, if WW' represents 250 lb., then WR and WS represent the required forces.)

4. Two forces, one of 100 lb. and the other of 75 lb., act on a body, one force pulling N. 20° E., the other N. 40° E. Find the resultant force.

HINT. Find the eastern and northern components of each force.

64. Slope of a straight line. Another important application of the trigonometric functions is in the study of the linear equation in x and y. We saw on page 29 that the graph of such an equation is a straight line, although we have not yet *proved* this to

be true. The question we shall now consider is the determination of the angle θ which a straight line makes with the X-axis. (By "the angle which one line (1) makes with another line (2)" we mean the *positive* angle through which (2) must rotate to come into coincidence with (1). Avoid the expression, "the angle *between* (1) and (2)," because that is ambiguous, there being always *two* angles between any two intersecting lines, unless the lines are perpendicular.)

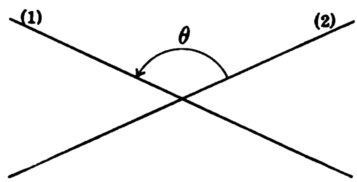


FIG. 52

65. Now, to find the angle which a given straight line[1] makes with the X-axis, take any two points P and Q on that line, and suppose their coördinates to be (x_1, y_1) and (x_2, y_2). Draw PM and QN parallel to the Y-axis, and PR parallel to the X-axis, thus obtaining the right triangle PRQ, in which the angle $RPQ = \theta$. Using this triangle,

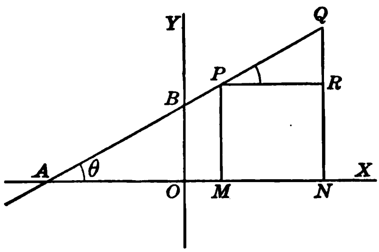


FIG. 53

$$\tan\theta = \frac{RQ}{PR}$$

and $RQ = NQ - NR = y_2 - y_1$,

while $$PR = MN = ON - OM = x_2 - x_1;$$

that is, $$\tan\theta = \frac{y_2 - y_1}{x_2 - x_1}. \qquad \text{(A)}$$

This simple result is of great importance, as we shall see. The truth of the equation (A) is not dependent upon which particular points P and Q we choose, on the line $ABPQ$. Thus, if we choose P and Q as in Fig. 54, the reasoning is as follows:

In the triangle PRQ, $\tan\theta = \dfrac{RQ}{PR}$

and $$RQ = NQ - NR = y_2 - y_1,$$

while $$PR = MO + ON = -x_1 + x_2 = x_2 - x_1,$$

[1] The given line is assumed not to be parallel to either coördinate axis.

the only difference being that PR is the *sum* of the absolute lengths of MO and ON, but x_1, the abscissa of P, is *not* MO, but OM[1]; that is, $MO = - x_1$.

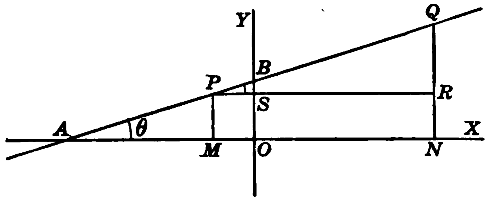


FIG. 54

Therefore

$$\frac{RQ}{PR} = \frac{y_2 - y_1}{x_2 - x_1},$$

so that formula (A) is true in this case also. The student should experiment further with points P and Q in various other positions on the line, and satisfy himself that in every case formula (A) is true.

When the angle θ is obtuse, the same result will again be found to hold. Thus, in Fig. 55, let P and Q be any two points on the given straight line, and let their coördinates be (x_1, y_1) and (x_2, y_2) respectively. Draw perpendiculars to the X-axis from P and Q, and the perpendicular to the Y-axis from Q, forming the triangle PRQ. Regarding Q as the vertex of the obtuse angle θ, the definition of $\tan\theta$ is

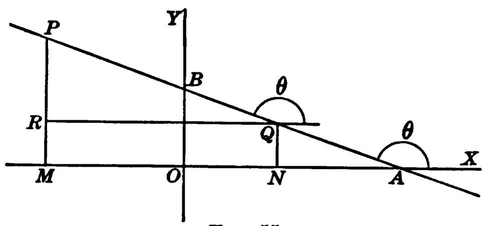


FIG. 55

$$\tan\theta = \frac{RP}{QR}\left(\text{observe that it is not } \frac{RP}{RQ}\right).$$

Now $\qquad RP = MP - MR = y_1 - y_2,$

and $\qquad QR = NO + OM = - x_2 + x_1$ (since $NO = - x_2$).

Therefore $\tan\theta = \dfrac{y_1 - y_2}{x_1 - x_2}$, which equals $\dfrac{y_2 - y_1}{x_2 - x_1}$.

Thus, formula (A) is correct in this case also.

If the line is parallel to the X-axis, we shall say $\theta = 0$, and formula (A) still holds true.

[1] A very common mistake in work of this kind is to write $OM = - x_1$, on account of the abscissa of P (or of M) being *negative*. *By definition* the abscissa *is* OM, and hence $MO = - x_1$.

In work of this kind it will be noticed that the all-important thing is to keep in mind the fact that every segment parallel to the X- or the Y-axis is a *directed* segment, and that abscissas are always measured *from the Y-axis*, ordinates *from the X-axis.*

66. The number $\tan\theta$ is called the *slope* of the line PQ. In words, the slope of a straight line is *the tangent of the angle which it makes with the X-axis.* We shall hereafter usually designate the slope by m. Formula (A) is then

$$\tan\theta = m = \text{slope of } PQ = \frac{y_2 - y_1}{x_2 - x_1},$$

or, in words, the slope of the straight line through two points is *the difference of their ordinates divided by the difference of their abscissas* (both differences being taken in the same order).

EXERCISES

Draw careful figures in all cases.

1. Find the slope of the line through the points $(2, 3)$ and $(5, 6)$; through the points $(3, 1)$ and $(-3, -1)$.

2. Find the slope of the line through the origin and the point $(4, 3)$; through the points $(-2, -3)$ and $(1, 0)$.

3. Find the slope of the line through $(2, 0)$ and $(0, -3)$; through $(a, 0)$ and $(0, b)$.

4. Write down the values of $\sin\theta$ and $\cos\theta$ in each of the Exs. 1–3 above.

5. What is the slope of the line $x - y = 2$? (In drawing the graph you necessarily get two points on the line; hence its slope can be found.)

6. What is the slope of the line $2x - 3y = 5$? of the line $x + 2y = 3$?

7. What is the slope of the line $x - ny = 5$? of the line $y = mx + k$? of the line $ax + by + c = 0$?

8. Find the angle which the line $4x + 6y = 7$ makes with the X-axis.

9. Prove that if two straight lines are perpendicular to each other, and if the slope of one is $\frac{2}{3}$, the slope of the other is $-\frac{3}{2}$.

10. Prove that if two straight lines are perpendicular to each other, and if the slope of one is m, that of the other is $-\dfrac{1}{m}$.

REVIEW PROBLEMS ON THE TRIGONOMETRIC FUNCTIONS

1. Prove that $\tan\theta + \cot\theta = \sec\theta \cdot \csc\theta$, θ being any angle.

Solution. By definition, $\cot\theta = \dfrac{1}{\tan\theta}$.

$$\text{Therefore } \tan\theta + \cot\theta = \tan\theta + \frac{1}{\tan\theta} = \frac{\tan^2\theta + 1}{\tan\theta}$$

$$= \frac{\sec^2\theta}{\tan\theta} = \sec\theta \cdot \frac{\sec\theta}{\tan\theta} = \sec\theta \cdot \frac{\frac{1}{\cos\theta}}{\frac{\sin\theta}{\cos\theta}}$$

$$= \sec\theta \cdot \frac{1}{\sin\theta} = \sec\theta \cdot \csc\theta. \qquad \text{Q.E.D.}$$

In problems 2–14, θ and α represent any angle. Prove:

2. $\sec\theta - \cos\theta = \sin\theta \cdot \tan\theta$.

3. $\dfrac{1 - \sin\theta}{\cos\theta} = \dfrac{\cos\theta}{1 + \sin\theta}$.

4. $\sin^4\theta - \cos^4\theta = \sin^2\theta - \cos^2\theta$.

5. $(\sin^2\alpha - \cos^2\alpha)^2 = 1 - 4\cancel{\sin^2\alpha\cos^2}\alpha$.

6. $\dfrac{1}{1 + \tan^2\alpha} + \dfrac{1}{1 + \cot^2\alpha} = 1$.

7. $(\sin\alpha + \cos\alpha)^2 + (\sin\alpha - \cos\alpha)^2 = 2$.

8. $\dfrac{\sec\theta - \csc\theta}{\sec\theta + \csc\theta} = \dfrac{\tan\theta - 1}{\tan\theta + 1}$.

9. $(\cot\theta + 2)(2\cot\theta + 1) = 2\csc^2\theta + 5\cot\theta$.

10. $\dfrac{\tan\theta - \sin\theta}{\sin^3\theta} = \dfrac{\sec\theta}{1 + \cos\theta}$.

11. $\dfrac{\tan\alpha - \cot\alpha}{\tan\alpha + \cot\alpha} = \dfrac{2}{\csc^2\alpha} - 1$.

12. $\sec^2\theta\csc^2\theta - 2 = \tan^2\theta + \cot^2\theta$.

13. $\cot\theta - \sec\theta\csc\theta(1 - 2\sin^2\theta) = \tan\theta$.

14. $(3\sin\alpha - 4\sin^3\alpha)^2 + (4\cos^3\alpha - 3\cos\alpha)^2 = 1$.

In each of the following problems, α and β are the acute angles of the right triangle ABC, and a, b, and c the sides.

15. Prove that $\tan \frac{\alpha}{2} = \frac{c-b}{a}$.

Solution. Bisect $\angle\alpha$ and draw the perpendicular from B upon this bisector, producing it to meet AC (produced) at F. Then $\triangle BCF$ is similar to $\triangle AEF$, and

$$\tan \frac{\alpha}{2} = \tan \angle FBC = \frac{CF}{BC} = \frac{c-b}{a}. \quad \text{Q.E.D.}$$

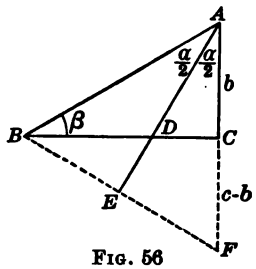


FIG. 56

Prove:

16. $\sin 2\alpha = \cos(\beta - \alpha)$. $(\beta > \alpha)$

17. $\cos 2\alpha = \frac{(b+a)(b-a)}{c^2}$.

18. $\tan 2\alpha = \frac{2ab}{(b+a)(b-a)}$.

19. $\cos(\beta - \alpha) = \frac{2ab}{c^2}$.

20. $\tan \frac{\alpha}{2} = \sqrt{\frac{c-b}{c+b}}$.

21. Find the angle which the line $3x - 4y = 12$ makes with the X-axis.

22. Find the angle which the line $x + y = 3$ makes with the line $3x - y = 5$. Find also the coördinates of the point of intersection of the two lines.

23. The equations of the sides of a triangle are $2x - 3y = 5$, $x + 5y = 9$, and $3x + 2y = 1$; find (a) the coördinates of the vertices, and (b) the angles of the triangle.

CHAPTER V

SOME SIMPLE FRACTIONAL AND IRRATIONAL FUNCTIONS; THE LOCUS PROBLEM

67. Review questions. What is the meaning of the statement "y is a function of x"? What is meant by the "graph of a function"? Classify functions according to degree. What is the nature of the graph of a function of the first degree? of the second degree? How can a quadratic equation be solved graphically? How can the nature of the roots of a quadratic equation be determined without solving it? Define the six trigonometric functions of an angle. State the most important relations among these functions. Define "slope of a straight line." When will a line have a negative slope?

68. Rational and irrational functions. The functions whose graphs we have constructed (Exercises, p. 22) involved only the operations of addition, subtraction, and multiplication, so far as the independent variable x was concerned. Such functions are called *integral rational functions.* The most general form of such a function of x is $f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n$. As in Exs. 10 and 13, p. 22, the *coefficients* of certain terms may contain fractions, but the function is still called *integral.* Thus, the quadratic function $y = ax^2 + bx + c$ is an integral rational function, whatever may be the values of a, b, and c.

If the independent variable occurs in the *denominator* of a fraction in its lowest terms (that is, if the operation of *division* has to be performed with the independent variable in the divisor), then the function is called a *fractional rational function,*— for example, $y = \frac{2}{x}$, $y = \frac{3+x}{x+1}$, $y = \frac{x^2}{1+x^2}$. On the other hand, $\frac{x}{2}$ and $\frac{x^2+3}{5}$ are *integral* rational functions of x.

Either of these two kinds of function is called a *rational function*, so that a rational function may be defined as one that involves any or all of the four fundamental operations,— addition, subtraction, multiplication, and division.

If the functional relation is such that a root must be extracted in order to arrive at the value of the function, it is said to be an *irrational function*,— for example, $y=\sqrt{x}$, $y=\sqrt{2x^2+1}$, $y=\sqrt{\frac{2}{x-3}}$, $y=\sqrt[3]{x^2-5x}$, $y=\sqrt{x+\sqrt{x}}$.

69. Graphs of rational functions.

Example 1. Draw the graph of the function $\frac{1}{x}$. The table of corresponding values of x and y, when $y=\frac{1}{x}$, is as follows:

x	1	2	3	4	$\frac{1}{2}$	$\frac{1}{3}$	-1	-2	-3	$-\frac{1}{2}$	etc.
$y=f(x)$	1	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	2	3	-1	$-\frac{1}{2}$	$-\frac{1}{3}$	-2	

The value $x=0$, it should be observed, gives no value of y at all, since division by zero is impossible. This means that the graph does not meet the Y-axis, because, for points on the Y-axis, $x=0$. But for very small positive values of x, y is very large, so that the graph is very far above the X-axis when near the Y-axis in the first quadrant. For *negative* values of x near to 0 the value of y is very large numerically, but negative, so that the graph is very far *below* the X-axis when near the Y-axis in the third quadrant. Thus the curve is separated into two parts, or "branches," each of which approaches the Y-axis more and more closely as the numerical value of y increases.[1]

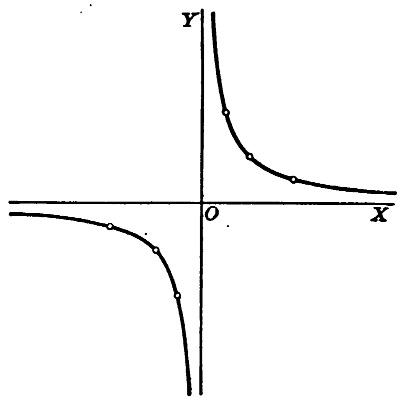


Fig. 57

When a curve continually approaches a straight line in such a way that as either x or y increases without limit the distance between the curve and

[1] What was just stated was of course the converse of this, namely, that when x decreases, y increases; but the converse statement is equally obvious from the equation $y=\frac{1}{x}$.

the line eventually becomes and remains less than any assignable value, the line is called an **asymptote** to the curve. Thus, the Y-axis is an asymptote to the graph of the function $\frac{1}{x}\cdot$ It can easily be seen by writing the equation in the form $x = \frac{1}{y}$ that the X-axis is also an asymptote to this curve, since, as x increases without limit, y decreases and becomes less than any assignable value. The curve is called a **hyperbola.**

Example 2. Draw the graph of the function $\frac{x+1}{x-2}\cdot$

The table of corresponding values of x and y, when $y = \frac{x+1}{x-2}$, is as follows:

x	0	1	2	3	4	5	6	-1	-2	-3	-4	$\frac{3}{2}$	$\frac{5}{2}$
y	$-\frac{1}{2}$	-2	No value	4	$\frac{5}{2}$	2	$\frac{7}{4}$	0	$\frac{1}{4}$	$\frac{2}{5}$	$\frac{1}{2}$	-5	7

Here the value $x = 0$ gives a definite value for $y\,(y = \frac{1}{2})$, but $x = 2$ makes the denominator zero and hence gives no value of y. As in the case of the Y-axis in the preceding example, so here the line $x = 2$ is easily seen to be an asymptote to the graph, because, for very large values of y, x is very near the value 2. There is also another asymptote, which can be found by letting x increase without limit. To do this it will be found convenient to change the form of the fraction as follows: divide both numerator and denominator by x, thus obtaining $y = \frac{x+1}{x-2} = \frac{1+\frac{1}{x}}{1-\frac{2}{x}}\cdot$ Now as x increases without limit, both $\frac{1}{x}$ and $\frac{2}{x}$ diminish and become eventually less than any assignable value. Thus, $\frac{x+1}{x-2} = \frac{1+\frac{1}{x}}{1-\frac{2}{x}}$ approaches 1, that is, y approaches 1. This means that $y = 1$ is an asymptote. This curve is also a hyperbola.

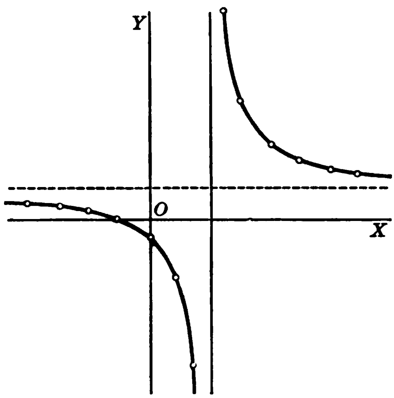


FIG. 58

(The fact that $y = 1$ is an asymptote could also have been discovered by solving the equation $y = \frac{x+1}{x-2}$ for x as a function of y, which gives $x = \frac{2y+1}{y-1}$, and the form of this fraction shows that $y = 1$ is an asymptote to the curve.)

EXERCISES

Draw the graphs of the following, showing the asymptotes if there are any:

1. $xy = 4.$
2. $xy = -10.$
3. $y = \dfrac{1}{x-4}.$
4. $y = \dfrac{1}{x+4}.$
5. $y = \dfrac{x+3}{x+5}.$
6. $y = \dfrac{x-2}{x-1}.$
7. $y = \dfrac{2x-1}{3x-1}.$
8. $y = \dfrac{2x+3}{3x-6}.$
9. $y = \dfrac{-x-1}{2x+5}.$
10. $y = \dfrac{x}{1-x}.$
11. $y = \dfrac{3x-4}{5-2x}.$
12. $y = \dfrac{7x-1}{x}.$
13. $y = \dfrac{x^2}{x-1}.$
14. $y = \dfrac{x^2-4}{x}.$
15. $y = \dfrac{2x^2-8}{3x+1}.$
16. $y = \dfrac{x}{x^2-1}.$
17. $y = \dfrac{x}{x^2+1}.$
18. $y = \dfrac{x-1}{x^2-4}.$
19. $y = \dfrac{x^2-2x-3}{x^2+x-2}.$

20. Prove that the graph of the function $y = \dfrac{ax+b}{cx+d}$ will have as asymptotes $x = -\dfrac{d}{c}$ and $y = \dfrac{a}{c}.$

70. Irrational functions. We shall consider only such irrational functions as involve *square roots.* Several examples of irrational functions will be given, in order to bring out all the details that must be attended to in the study of this class of functions.

Example 1. $f(x) = \pm\sqrt{1+x}.$

Write, as usual, $y = f(x)$, that is, $y = \pm\sqrt{1+x}$. For any value of x that is < -1, y will be complex; hence no point to the left of $x = -1$ can be found upon the graph. For $x = -1$, $y = 0$; and for any value of x which is > -1, y will have two values that are equal numerically but of opposite sign, as in the following table:

x	-1	0	1	2	3	4	5	6	7	8	
y	0	± 1	$\pm\sqrt{2}$	$\pm\sqrt{3}$	± 2	$\pm\sqrt{5}$	$\pm\sqrt{6}$	$\pm\sqrt{7}$	$\pm\sqrt{8}$	± 3	etc.

Plotting these points and joining them by a smooth curve, we have the graph of the function (Fig. 59). The curve is symmetrical with respect to

the X-axis because of the fact just noted, that for every value of $x\ (> -1)$ y has *two* values which are numerically equal but of opposite sign, thus giving two points symmetrically located with regard to the X-axis. The curve is a parabola.

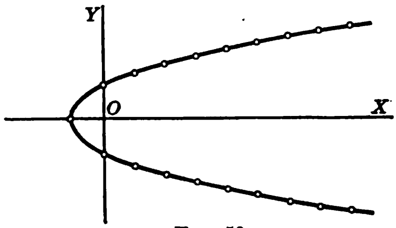


FIG. 59

The functional relation of this example can be written in the form $y^2 = 1 + x$, or $y^2 - x = 1$, or $y^2 - x - 1 = 0$. In any of these forms y is said to be an *implicit* function of x, because the functional relation is definitely implied by the equation; when the equation is solved for y, y is said to be an *explicit* function of x.

Example 2. $f(x) = \pm\sqrt{1 - x^2}$.

Writing $y = f(x)$, we see that y is complex if $x^2 > 1$, that is, if $x > +1$ or < -1. Hence the only values of x that will give points on the graph are those between -1 and $+1$ (inclusive). The table of values is as follows:

x	-1	0	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	1	(any negative value > -1)
y	0	± 1	$\pm\sqrt{\frac{15}{16}}$	$\pm\sqrt{\frac{3}{4}}$	$\pm\sqrt{\frac{7}{16}}$	0	(same as for corresponding + value)

Plotting these points and joining them by a smooth curve, we find that the graph seems to be a circle (Fig. 60). This is in fact the case, as will be shown later. y is given as an implicit function of x by the equation $y^2 = 1 - x^2$, or $x^2 + y^2 = 1$.

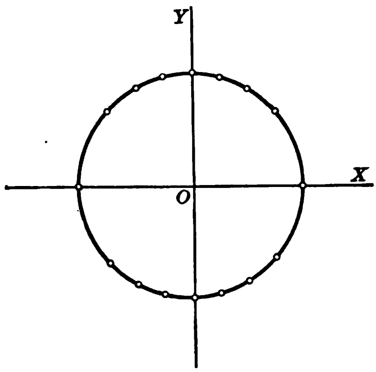


FIG. 60

Example 3. $f(x) = \pm\sqrt{4 - 4x^2}$.

If, as usual, $f(x)$ is represented by y, then the equation can be written in the implicit form

$$4x^2 + y^2 = 4.$$

Within what limits must x lie in order that y may have real values? Having answered this question, the table of corresponding values of x and y should be drawn up, as follows:

x	0	$\pm\frac{1}{2}$	$\pm\frac{1}{4}$	$\pm\frac{3}{4}$	± 1
y	± 2	$\pm\sqrt{3}$	$\pm\sqrt{\frac{15}{4}}$	$\pm\sqrt{\frac{7}{4}}$	0

Joining the points by a smooth curve, we have the graph of this function as in Fig. 61. It is a closed curve, symmetrical with respect to each of the coördinate axes. The curve is called an *ellipse.*

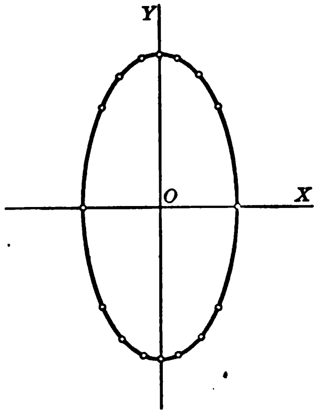


FIG. 61

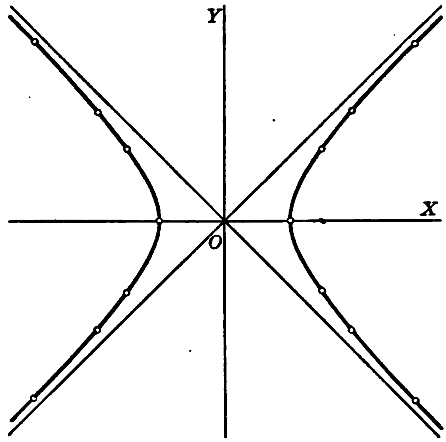


FIG. 62

Example 4. $f(x) = \pm\sqrt{x^2-1}$, or $x^2 - y^2 = 1$, when written in the implicit form. What are the limitations on the values of x here? The table of values is as follows:

x	± 1	± 2	± 3	± 4	$\pm \frac{3}{2}$
y	0	$\pm\sqrt{3}$	$\pm\sqrt{8}$	$\pm\sqrt{15}$	$\pm\sqrt{\frac{5}{4}}$

The graph of this function consists of two parts, symmetrically located with respect to the Y-axis, each part being symmetrical with respect to the X-axis. The curve is a hyperbola; its similarity in form to the graph of $y = \frac{1}{x}$ is evident, the only difference being in the *position* of the curve. A characteristic feature of the hyperbola is the presence of two asymptotes. In the case of $y = \frac{1}{x}$ they were the X-axis and the Y-axis, while here they are the lines through the origin with slopes $+1$ and -1 respectively. The distances from points on the curve to either line become and remain less than any positive distance that can be mentioned, as the values of x or y increase without limit. This assertion will be proved later (see p. 134). Fig. 62 shows the asymptotes as well as the curve.

EXERCISES

Make a careful graph of each of the functions of x given by the following equations, and state in every case for what values of x, if any, the function fails to have a value. If the graph is a hyperbola, draw the asymptotes, as near as you can get them.

1. $x^2+3y^2=12$.	8. $2x^2+5y^2=5$.	15. $2x^2-3y^2+5=0$.
2. $x^2-2y^2=4$.	9. $y^2+x=4$.	16. $4x^2-9y^2=36$.
3. $x^2+4y^2=4$.	10. $4y^2-x=8$.	17. $4x^2+9y^2=36$.
4. $y^2-4x=4$.	11. $7x^2+y^2=28$.	18. $4x-9y^2=36$.
5. $x^2+y^2=9$.	12. $4y^2-x^2=4$.	19. $4x+9y^2=36$.
6. $3x^2+y^2=6$.	13. $4x^2-y^2=4$.	20. $4x^2-9y^2=-36$.
7. $y^2-x^2=1$.	14. $x^2+3y^2=5$.	21. $2x^2+2y^2=11$.

THE LOCUS PROBLEM

71. One of the most important uses of the graphical representation of functions is in the study of geometric *loci*. The word "locus" has already been used (p. 19), but before going farther it will be well to review carefully a few examples which will aid in making the precise meaning of the word clear.

1. The locus of points equidistant from two fixed points is the perpendicular bisector of the line segment joining the two points.

2. The locus of points equidistant from two fixed intersecting straight lines is the *bisectors* of the angles formed by the lines.

3. The locus of points at a constant distance from a fixed point is a circle about the fixed point as center, and with the constant distance as radius.

72. Definition of locus. As these illustrations remind us, the locus of points that satisfy a certain condition means the *totality* of points satisfying that condition. The locus must, first, contain all the points that satisfy the condition, and, secondly, must not contain any point that fails to satisfy the condition. No statement about a "locus of points" can be justified unless it can be shown that both these things are true of the alleged locus. Each of the three examples given above should be carefully tested to see if

it fulfills both requirements of a locus. Notice that 2 is not true if the italicized word *bisectors* be changed to *bisector*. Notice also that 1, 2, and 3 are not true if we consider points outside of the plane. What is the corresponding locus in *space* in each case?

73. It was stated above that the graphic representation can be used in solving problems concerning loci. This fact has been partly brought out in the work at the beginning of Chap. II, p. 19, and those examples will now be considered again.

Example 1. The locus of points at the distance 2 from the X-axis is evidently the straight line parallel to the X-axis and 2 units distant from it, because, first, all points on the line ABC (Fig. 15, p. 19) are at the distance 2 from the X-axis, and, secondly, every point that is at the distance 2 from the X-axis lies on this line, since a point *above* the line ABC will be at a distance *greater* than 2 from the X-axis and a point *below* the line ABC will be at a distance *less* than 2 from the X-axis. (Of course, if we were not dealing with *directed* distances, the locus would consist of *two* straight lines, one above and one below the X-axis; but the latter is the locus of points whose distance from the X-axis is -2.)

Example 2. The locus of points that are twice as far from the X-axis as from the Y-axis is a straight line through the origin, with slope 2.

The *result* of this problem was stated on page 20, and indeed it is nearly self-evident that this locus is a straight line; but it is important even in these simple cases to make sure that the essential nature of the locus problem is not lost sight of. To *prove* that the line $P'OP$ (Fig. 16, p. 20) is the locus of points that are twice as far from the X-axis as from the Y-axis, it is necessary to show, first, that every point on this line satisfies the condition, and, secondly, that every point that satisfies the condition is on the line. First, if P is any point on the line, and if OM is the abscissa and MP the ordinate of P, then $\frac{MP}{OM} = \tan\theta = 2$, by hypothesis; that is, $MP = 2\,OM$, which means that P is in fact twice as far from the X-axis as from the Y-axis. Secondly, let P' (Fig. 63) be any point that is twice as far from the X-axis as from the Y-axis. Then $M'P' = 2\,OM'$; hence triangle $OM'P'$ is similar to triangle OMP; and

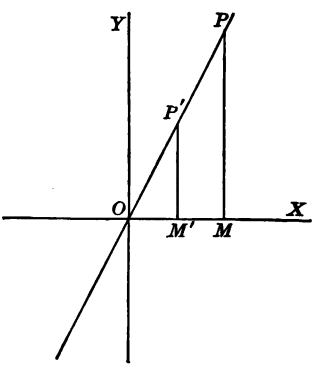


FIG. 63

therefore $\angle M'OP' = \angle MOP$. Therefore P' is on the line OP. This completes the proof of the theorem as stated.

It was seen on page 20 that the *equation* of this locus is $y = 2x$; hence we have now proved that the graph of the equation $y = 2x$ is the straight line through the origin with slope 2.

Example 3. What is the locus of points at a constant distance of 3 units from a fixed point?

Let the fixed point be chosen as origin, and let (x, y) be the coördinates of any point P on the locus. Then the geometric condition of the problem is

$$OP = 3. \qquad (1)$$

Now, by the Pythagorean Theorem,

$$OP = \sqrt{x^2 + y^2}. \qquad (2)$$

Hence $$\sqrt{x^2 + y^2} = 3. \qquad (3)$$

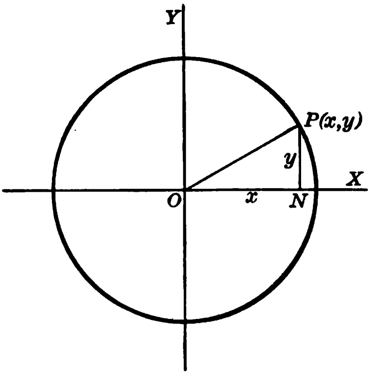


FIG. 64

This algebraic condition is thus equivalent to the geometric condition of the problem, so that the graph of equation (3) is the locus required. (In this case the locus is a circle, from the very definition of that curve. The same reasoning proves that the graph of Example 2, p. 89 (Fig. 60), was a circle, as was there asserted. The student should state the locus problem corresponding to that example.)

74. These examples illustrate the process of obtaining equations corresponding to geometric locus problems. This process may be summarized thus: First, represent by the variables (x, y) the coördinates of any point on the required locus (the X- and Y-axes being chosen in any convenient position). Secondly, translate the geometric condition of the problem into an algebraic condition or equation involving x and y, or either one of them alone. The graph of this equation will then be the locus required. The equation is called "the equation of the locus."

To be sure, no advantage of this process is evident in cases like those just given, where the locus was easily obtained by the application of well-known theorems of elementary geometry. But in many cases it is difficult, perhaps even impossible, to gain definite knowledge of the locus directly; whereas the equation of the locus *can* be obtained, and then the graph of this equation,

found by plotting points whose coördinates satisfy the equation, will be the desired locus. In later chapters we shall see that many important facts about a locus can often be established by drawing conclusions from the *equation* of the locus. For the present, however, the derivation of the equation is the main thing.

EXERCISES

Draw a figure for each.

1. Find the equation of the locus of points at a distance 2 from the Y-axis; from the X-axis.

2. Find the equation of the locus of points at a distance -5 from the X-axis.

3. What is the equation of the X-axis? of the Y-axis?

4. Find the equation of the locus of points that are 3 times as far from the Y-axis as from the X-axis. What is the locus? Prove the fact.

5. Find the equation of the locus of points that are -2 times as far from the X-axis as from the Y-axis. What is the locus? Give proof.

6. Find the equation of the locus of points at a distance c from the Y-axis.

7. Find the equation of the locus of points that are at a constant distance equal to 5 from the origin.

8. Find the equation of the locus of a point that moves so as to be always equidistant from the points (2, 1) and (3, 5).

Solution. Let $A \equiv (2, 1)$ and $B \equiv (3, 5)$; also let $P \equiv (x, y)$ be any point on the required locus. Then the condition of the problem is $PA = PB$. Each of these distances can be expressed algebraically by means of the result of Ex. 20, p. 12, which proved that the distance between two points (x_1, y_1) and (x_2, y_2) is

$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

Therefore

$$PA = \sqrt{(x-2)^2 + (y-1)^2}$$

and $$PB = \sqrt{(x-3)^2 + (y-5)^2}.$$

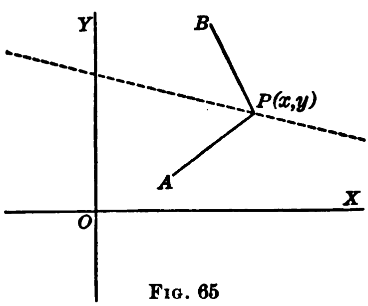


FIG. 65

Since $PA = PB$, the corresponding algebraic equation is

$$\sqrt{(x-2)^2 + (y-1)^2} = \sqrt{(x-3)^2 + (y-5)^2},$$

or, simplifying, $$2x + 8y = 29.$$

This is accordingly the equation of the locus. The graph is found to be a straight line, which, as we know, is the perpendicular bisector of the line segment AB. (Of course the mid-point of the segment AB must lie on the locus; its coördinates are $\left(\frac{2+3}{2}, \frac{1+5}{2}\right)$, that is, $\left(\frac{5}{2}, 3\right)$, and these coördinates must therefore satisfy the equation of the locus, $2x + 8y = 29$; in fact, $2 \cdot \frac{5}{2} + 8 \cdot 3 = 5 + 24 = 29$, checking partially the accuracy of the result obtained.)

9. Find the equation of the locus of a point that moves so as to be equally distant from the points (2, 3) and (5, 1).

10. Find the equation of the locus of points equidistant

(a) from the points (1, 3) and (– 1, 4);
(b) from the points (3, 2) and (– 5, – 1);
(c) from the points (– 1, – 2) and (3, – 5);
(d) from the points (2, 2) and (0, 0);
(e) from the points (2, 0) and (0, 2);
(f) from the points (0, – 4) and (– 6, 0);
(g) from the points (a, 0) and (0, b).

11. A point moves so that its distance from the origin is constantly equal to 10. Find the equation of its locus.

12. A point moves so that its distance from the point (2, 4) is always equal to 6. Find the equation of its locus.

13. Find the equation of the path of a point that moves so as to be equally distant from the points (– 2, – 3) and (2, – 5).

14. A point moves so that its distance from the point (2, 0) is always equal to its distance from the Y-axis. Find the equation of its locus.

Solution. Let $P \equiv (x, y)$ be any position of the moving point (that is, any point on the required locus). Then the condition of the problem is $AP = BP$ (Fig. 66). The algebraic equivalents of these geometrical quantities are

$$AP = \sqrt{(x-2)^2 + y^2}$$

and $$BP = x.$$

Hence the required equation is

$$\sqrt{(x-2)^2 + y^2} = x.$$

That is, $$x^2 - 4x + 4 + y^2 = x^2,$$

or $$y^2 = 4x - 4,$$

or $$x = \frac{y^2 + 4}{4}.$$

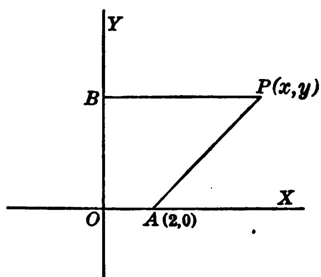


FIG. 66

This is therefore the equation of the required locus. Here we have an illustration of a problem whose solution could not be obtained by using any of the theorems of elementary geometry merely, but by drawing a careful graph of the equation of the locus we can gain a very good idea of the appearance of the locus itself. It will be found to be a parabola.

15. A point moves so that its distance from the X-axis is always equal to its distance from the point $(0, 4)$. Find the equation of its locus.

16. Find the equation of the locus of a point which moves so that its distance from the point $(6, 1)$ is always equal to its distance from the line $x = 2$.

In each of the following problems, find the equation of the locus of a point that moves according to the condition given, and draw the graph of the equation:

17. Its distance from the point $(-2, 5)$ is always equal to its distance from the line $y = -3$.

18. Its distance from the point $(4, 0)$ is always twice its distance from the line $x = 2$.

19. Its distance from the point $(5, 3)$ is always one half its distance from the line $y = 1$.

20. Its distance from the point $(-1, 4)$ is always one half its distance from the Y-axis.

21. Its distance from the point $(3, 0)$ is always two thirds its distance from the line $x = 4$.

22. The sum of its distances from the points $(2, 0)$ and $(-2, 0)$ equals 6.

23. The difference of its distances from the points $(2, 0)$ and $(-2, 0)$ equals 3.

24. Its distance from $(3, 0)$ is constantly equal to twice its distance from $(-3, 0)$.

25. The sum of the squares of its distances from the points $(3, -2)$ and $(-3, -4)$ is always equal to 70.

26. The difference of the squares of its distances from the points $(2, 1)$ and $(-3, -5)$ is always equal to 19.

27. The square of its distance from the origin is always equal to the sum of its distances from the X-axis and the Y-axis.

CHAPTER VI

THE STRAIGHT LINE AND THE CIRCLE

75. Having become somewhat familiar with the locus problem in a general way, we now take up the study of the simpler types of loci in detail, in order to learn the great power of the methods of algebra as applied to geometric problems. We shall see that to certain types of equations between x and y correspond certain loci, the connection being so simple that we can tell at a glance what the locus of a given equation will be.

Example 2, p. 92, proved that the locus of points twice as far from the X-axis as from the Y-axis corresponds to the equation $y = 2x$, and that the graph is the straight line through the origin with slope 2. In the same way we are led to the general theorem:

The equation of the straight line through the origin with slope m is $y = mx$.

The proof will be given, although it is practically a repetition of the work of the problem just referred to. We must show, first, that if $P \equiv (x, y)$ is any point on the straight line AOB (the straight line through the origin with slope m), then the coördinates (x, y) of P satisfy the equation $y = mx$; and, secondly, that if the coördinates of any point $P' \equiv (x', y')$ satisfy the equation $y = mx$, then the point P' lies on the line AOB.

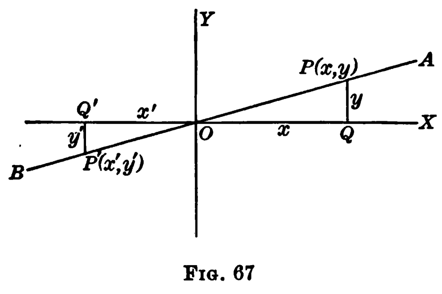


FIG. 67

Proof. First, if P is any point on the line AOB (Fig. 67), then

$$\frac{y}{x} = \tan \angle XOA = m;$$

that is,
$$y = mx.$$

Secondly, if the coördinates of P' (x' and y') satisfy the equation $y = mx$, then

$$\frac{y'}{x'} = m;$$

that is, $$\tan \angle XOP' = \tan \angle XOA.$$

Therefore $\angle XOP' = \angle XOA$ or else $180° + \angle XOA$, and in either case the point P' is on the straight line AOB. This completes the proof that the equation $y = mx$ is the equation of the straight line through the origin with slope m.

EXERCISES

Write down the equations of the straight lines through the origin with each of the following slopes: $1, -2, \frac{2}{3}, \frac{5}{2}, -5, 7, -1, -\frac{1}{2}, \sqrt{2}, -\frac{\sqrt{3}}{3}, \frac{3}{4}$. Draw a figure for each.

76. As our next problem, let us find the equation of the straight line through the point (1, 3) with the slope $\frac{1}{2}$.

If $P \equiv (x, y)$ is any point on the straight line (Fig. 68), then, by theorem (A) on page 80, the slope of the line is

$$\frac{y-3}{x-1}.$$

Since the slope is equal to $\frac{1}{2}$,

$$\frac{y-3}{x-1} = \frac{1}{2}.$$

That is, $$2y - 6 = x - 1,$$

or $$x - 2y + 5 = 0,$$

which is accordingly the equation of the line AP.

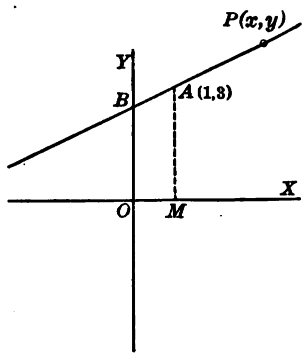


Fig. 68

EXERCISES

1. Find the equation of the line

   (a) through the point (2, − 3) with slope 1;
   (b) through the point (0, 3) with slope 2;
   (c) through the point (−1, − 2) with slope $\frac{3}{2}$;
   (d) through the point (7, 1) with slope − 3;
   (e) through the point (− 2, − 3) with slope $-\frac{1}{2}$.

2. Find the equation of the straight line through the point $(0, k)$ with slope m.

Ans. $\boldsymbol{y = mx + k}$. This equation is called the *slope-intercept* form of equation of the straight line, because it shows at a glance the slope (m) and the Y-intercept (k) of the line (cf. p. 29). Any equation of a straight line can be reduced to this form (if y is present in the equation) by solving for y. Thus, $3x - y = 2$ is equivalent to $y = 3x - 2$, in which form $m = 3$ and $k = -2$.

3. By reducing each of the following equations to the slope-intercept form, read off the value of the slope and the Y-intercept. Verify by the graph, as usual: $x + y = 2$, $3x + 4y = 5$, $7x - 3y = 1$, $\frac{1}{2}x + \frac{3}{4}y = 3$, $lx + y = b$, $x - ny = q$.

4. Find the equation of the line through the point $(-3, -5)$ with slope m.

5. Find the equation of the line through the point (x_1, y_1) with slope m.

$$\textit{Ans.}\quad \frac{y - y_1}{x - x_1} = m, \quad \text{or} \quad \boldsymbol{y - y_1 = m(x - x_1)}.$$

This is called the *slope-point* form of equation of a line.

6. Write down the equation of the line through the point $(5, 3)$ with slope 2.

7. Write down the equation of the line through the point $(-1, 2)$ with slope -1.

8. Write down the equation of the line through the point $(2, -3)$, parallel to the line $y = \frac{2}{3}x + 6$.

9. Find the equation of the line through the origin, parallel to the line $x + 2y = 3$.

10. Find the equation of the line through the point $(1, -1)$, parallel to the line $\frac{1}{2}x - 3y = 6$.

11. Find the equation of the line through the point $(-5, 1)$, perpendicular to the line $2x - 3y = 4$.

HINT. The slope of the required line can be found from that of the given line by means of Ex. 10, p. 83.

12. Find the equation of the line through the point $(1, 4)$, perpendicular to the line $x - 3y = 10$.

13. Find the equation of the line through the origin, perpendicular to the line $x + y = 4$.

14. Find the equation of the straight line through the points (3, 4) and (4, 1).

HINT. Find the slope of the line by the theorem on page 80, then follow the method of the preceding exercises.

15. Find the equation of the line through the points

(a) (4, 1) and (− 1, − 5); (c) (4, 3) and (4, − 1);
(b) (4, − 4) and (3, 2); (d) (0, 0) and (− 5, − 3);
(e) (3, 4) and (− 2, 4);
(f) (x_1, y_1) and (x_2, y_2).

Ans. $\dfrac{y-y_1}{x-x_1}=\dfrac{y_2-y_1}{x_2-x_1}$, or $\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x & y & 1 \end{vmatrix}=0.$

16. Find the equation of the line whose X- and Y-intercepts are a and b respectively. *Ans.* $\dfrac{x}{a}+\dfrac{y}{b}=1.$

17. Find the equation of the line through the point (− 2, 3) and making an angle of 135° with the X-axis.

18. Find the equation of the line through the point (1, 2) and making an angle of 60° with the X-axis.

77. These exercises illustrate the fact that the equation of a straight line can be found if its slope and a point on it are given. Ex. 5 brought this out definitely, and it is of fundamental importance in the study of straight lines. We can now state as a theorem what we have hitherto tacitly assumed: *The equation of any straight line is of the first degree in x and y.*

For every straight line has a definite slope (unless it is perpendicular to the X-axis, in which case its equation is $x = k$, which is of the first degree), and hence its equation is $y - y_1 = m(x - x_1)$, where (x_1, y_1) is any fixed point on the line. But this equation is of the first degree; hence the theorem is proved.

78. Conversely, *any equation of the first degree in x and y has for its graph a straight line.*

Proof. The general equation of the first degree in x and y can be written

$$ax + by + c = 0.$$

If $b \neq 0$, we can divide by b, getting

$$\frac{a}{b}x + y + \frac{c}{b} = 0;$$

that is,

$$y = -\frac{a}{b}x - \frac{c}{b}.$$

This is now in the form $y = mx + k$, where $m = -\frac{a}{b}$ and $k = -\frac{c}{b}$. But Ex. 2 above proved that this is the equation of the straight line whose slope is m, that is, $-\frac{a}{b}$, and whose Y-intercept is k, that is, $-\frac{c}{b}$. Hence the equation $ax + by + c = 0$ represents a straight line whose slope is $-\frac{a}{b}$ and whose Y-intercept is $-\frac{c}{b}$, provided $b \neq 0$.

But if $b = 0$, the equation is simply

$$ax + c = 0;$$

that is,

$$x = -\frac{c}{a}.$$

(Naturally a cannot equal 0, else the equation would not have been of the first degree.) This is the equation of a line parallel to the Y-axis, so that the theorem is proved also for the case $b = 0$.

Note that if $a = 0$, the line is parallel to the X-axis. Moreover, since $m = -\frac{a}{b}$, $m = 0$ in this case, so that lines parallel to the X-axis have the slope zero. This of course results also from the definition of "slope," as we observed on page 81.

79. Normal form of the equation of a straight line. Let AB be any straight line not passing through the origin, and ON the perpendicular from the origin upon the line. Let the length of ON be denoted by p, and the angle XON by ω. We shall, moreover, consider ON as being a *directed* line, the positive direction being *from*

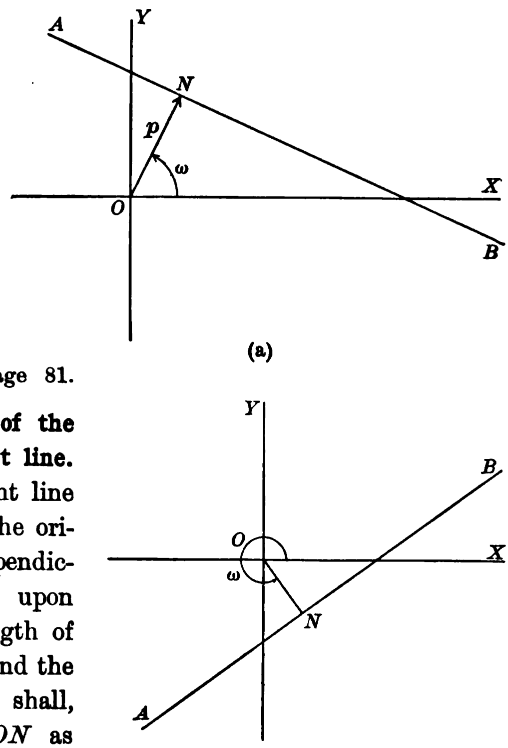


FIG. 69

O to N, in whatever position AB may lie. The angle ω may have any value from 0° to 360° ($0 \leqq \omega < 360$). The line ON is called the *normal* to the line AB.

In case we have to do with a line AB through the origin, the perpendicular to AB through O (OC in Fig. 70) is still called the normal to AB, and it is directed so that the angle ω is between 0° and 180° ($0 \leqq \omega < 180$).

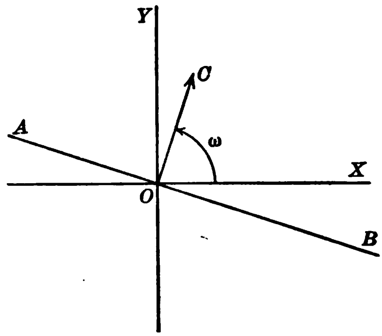


FIG. 70

Evidently, under these stipulations, any line AB determines a pair of values ω and p; and, conversely, any pair of values of ω and p determines a straight line.

EXERCISES

Construct the lines for which

(1) $\omega = 30°$, $p = 3$; (4) $\omega = 200°$, $p = \frac{4}{3}$; (7) $\omega = 180°$, $p = 4$;
(2) $\omega = 155°$, $p = 1$; (5) $\omega = 90°$, $p = 5$; (8) $\omega = 0°$, $p = 0$.
(3) $\omega = 330°$, $p = 2$; (6) $\omega = 100°$, $p = 0$; (9) $\omega = 90°$, $p = 0$.

Since ω and p determine the straight line AB completely, it must be possible to express the equation of AB in terms of these values. This is easily done by considering that the line AB passes through the point N (Fig. 71), and that its slope is the negative reciprocal of the slope of ON; that is, it equals $-\dfrac{1}{\tan \omega}$, which equals $-\dfrac{\cos \omega}{\sin \omega}$. Hence the equation of AB is

$$y - y_1 = -\frac{\cos \omega}{\sin \omega}(x - x_1). \quad (1)$$

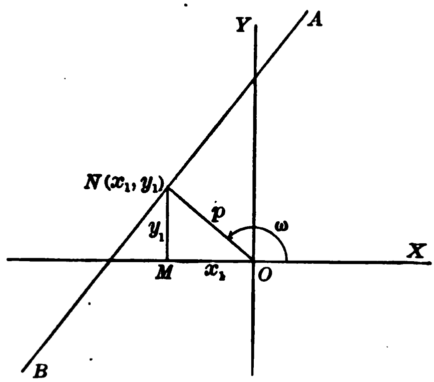


FIG. 71

But $$y_1 = MN \text{ (Fig. 71)} = p \sin \omega,$$
and $$x_1 = OM = p \cos \omega.$$

(1) is accordingly the same as
$$y - p \sin \omega = -\frac{\cos \omega}{\sin \omega}(x - p \cos \omega);$$
that is, $$x \cos \omega + y \sin \omega - p(\sin^2 \omega + \cos^2 \omega) = 0.$$

Since $$\sin^2 \omega + \cos^2 \omega = 1,$$
the equation takes the simpler form
$$\boldsymbol{x \cos \omega + y \sin \omega - p = 0,} \tag{2}$$
which is known as the "normal form of the equation of a straight line." Write in this form the equation of each of the nine lines in the above exercises.

80. Reduction of general linear equation to normal form. Any equation of a straight line can be reduced to the form (2). For example, suppose we have the equation $3x - 4y = 5$, and that we wish to reduce it to the form (2). The only change that we can make in an equation that will not change its graph is to add the same quantity to both sides of the equation or to multiply both sides by the same constant quantity. Since the right-hand side of (2) is 0, we first write our equation so that its right-hand side becomes 0:
$$3x - 4y - 5 = 0. \tag{3}$$
The only other change that can be made is to multiply both sides by the same number, say k;[1] that is, to write (3) in the form
$$3kx - 4ky - 5k = 0. \tag{4}$$

[1] A precise statement of the fact used here is, *If*
$$a_1x + b_1y + c_1 = 0 \tag{1}$$
represents the same line as $$a_2x + b_2y + c_2 = 0, \tag{2}$$
then $a_2 = ka_1$, $b_2 = kb_1$, *and* $c_2 = kc_1$; or (2) can be obtained from (1) by multiplying it by a constant k. This theorem may be proved as follows: The slope of (1) is $-\frac{a_1}{b_1}$, and that of (2) is $-\frac{a_2}{b_2}$. Hence, if the lines are the same, $\frac{a_1}{b_1} = \frac{a_2}{b_2}$; that is, $\frac{b_2}{b_1} = \frac{a_2}{a_1}$. Also, the Y-intercept of (1) is $-\frac{c_1}{b_1}$, while that of (2) is $-\frac{c_2}{b_2}$. Hence, if the lines are the same, $\frac{c_1}{b_1} = \frac{c_2}{b_2}$; that is, $\frac{b_2}{b_1} = \frac{c_2}{c_1}$. Therefore $\frac{a_2}{a_1} = \frac{b_2}{b_1} = \frac{c_2}{c_1}$. Call the value of this common ratio k, and we see that $a_2 = ka_1$, $b_2 = kb_1$, and $c_2 = kc_1$. Q.E.D.

If (2) is the same line as (4), k must have such a value as to make each term of (4) equal to the corresponding term of (2); that is,

$$\cos\omega = 3k, \quad \sin\omega = -4k, \quad p = 5k.$$

Therefore $\qquad \cos^2\omega + \sin^2\omega = 9k^2 + 16k^2 = 25k^2.$

But $\qquad \cos^2\omega + \sin^2\omega = 1.$

Therefore $\qquad k^2 = \frac{1}{25},$

$$k = \pm\tfrac{1}{5};$$

and $\qquad p = 5k = +1,$

the sign being determined by the fact that p is always positive. Hence only the + sign may be chosen for k.

Therefore $\qquad \cos\omega = \frac{3}{5}, \quad \sin\omega = -\frac{4}{5}, \quad p = 1,$

and the normal form of the equation $3x - 4y = 5$ is accordingly

$$\tfrac{3}{5}x - \tfrac{4}{5}y - 1 = 0. \tag{5}$$

By inspection of equation (5) we see that the length of the normal is 1, and that it makes with the X-axis an angle whose sine is $-\frac{4}{5}$ and whose cosine is $\frac{3}{5}$; that is, an angle in the fourth quadrant.

General case. The general case is treated in a similar way. Let the equation of a straight line be $ax + by + c = 0$, and let it be required to reduce this equation to the form (2) of § 79.

If (2) and $ax + by + c = 0$ represent the same line, we must be able to get the one equation from the other by multiplication by a constant k. The equation

$$akx + bky + ck = 0$$

must then be exactly the same as

$$x\cos\omega + y\sin\omega - p = 0.$$

Therefore $\qquad \cos\omega = ak, \quad \sin\omega = bk, \quad p = -ck.$

Therefore $\qquad \cos^2\omega + \sin^2\omega = (a^2 + b^2)k^2.$

Therefore $\qquad k^2 = \dfrac{1}{a^2 + b^2}$

and $\qquad k = \dfrac{1}{\pm\sqrt{a^2 + b^2}}.$

Since p is positive and equals $-ck$, the sign of k must be *opposite to that of c.* If $c = 0$, the angle ω is between 0° and

180° (§ 79); hence $\sin \omega$ is positive, so that the sign of k must be taken *the same as that of b.*

81. This result may be summarized as follows: *To reduce the equation of a straight line $ax + by + c = 0$ to the normal form, divide each term by $\pm\sqrt{a^2 + b^2}$, choosing the sign of the radical opposite to that of c, or, if $c = 0$, choosing the same sign as that of b.*

EXERCISES

1. Reduce each of the following equations to the normal form: $5x + 12y = 1$, $x + \frac{4}{3}y + 5 = 0$, $x + y + 1 = 0$, $2x - 3y - 8 = 0$, $3x + y = 4$. Draw a figure for each.

2. Find the distance between the parallel lines $x + 3y - 3 = 0$ and $2x + 6y + 1 = 0$.

3. Find the equation of a line parallel to the line $3x - 4y = 5$ and at a distance 3 from it. (Two solutions.)

82. Distance from a line to a point. One of the most important uses of the normal form of the equation of a straight line

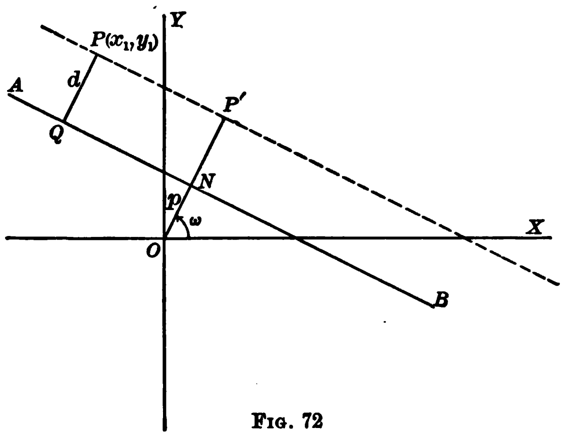


FIG. 72

is to determine the distance from a given line to a given point. Let the equation of the given line AB be

$$x \cos \omega + y \sin \omega - p = 0. \tag{1}$$

Let $P \equiv (x_1, y_1)$ be the given point, and d its distance from AB; draw through P the line PP' parallel to AB; the distance from

the origin to PP' will then equal $p + d$. The equation of PP' is accordingly

$$x \cos \omega + y \sin \omega - (p + d) = 0. \tag{2}$$

Since P is on this line, its coördinates (x_1, y_1) must satisfy the equation (2); that is,

$$x_1 \cos \omega + y_1 \sin \omega - p - d = 0.$$

Therefore

$$\boldsymbol{d = x_1 \cos \omega + y_1 \sin \omega - p}. \tag{3}$$

The result (3) may be stated in words thus: *In the normal form of the equation of the given line, substitute for x and y the coördinates x_1 and y_1 of the given point; the result will be the distance from the given line to the given point.*

If the equation of the given line is in the form $ax + by + c = 0$, it must be reduced to the normal form

$$\frac{ax}{\pm\sqrt{a^2+b^2}} + \frac{by}{\pm\sqrt{a^2+b^2}} + \frac{c}{\pm\sqrt{a^2+b^2}} = 0.$$

Then, substituting the coördinates of the given point (x_1, y_1) for (x, y), we have

$$\boldsymbol{d = \frac{ax_1 + by_1 + c}{\pm\sqrt{a^2+b^2}}}, \tag{4}$$

the sign of the radical being opposite to that of c.

83. In Fig. 72 the point P was on the opposite side of the line from the origin, so that the direction *from the line AB to the point P* was the same as the positive direction of the normal ON. Hence d, that is, QP, may also be considered as *positive*, while in case P were on the *same* side of AB as the origin (as in Fig. 73), the direction from the line AB to the point P would be *opposite* to the positive direction of the normal ON. Hence d may in this case be considered negative.

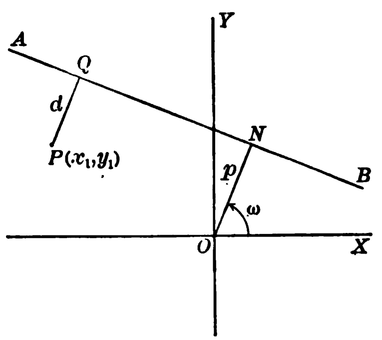


FIG. 73

Thus, for all points P on the opposite side of AB from the origin, d would be positive, and for all points P on the same side

of AB as the origin, d would be negative. The student should now verify the fact that the formula (3) above gives the *sign* of d correctly according to this agreement. Every straight line thus divides the plane into two parts, which may be called the positive and negative sides of the line. The origin is, then, always on the negative side of any line (unless the line passes *through* the origin, in which case the upper side of the line is the positive side) (see Fig. 70).

EXERCISES

1. How far is the point (1, 4) from the line $x + y + 1 = 0$?

Solution. Using the formula (4),

$$d = \frac{ax_1 + by_1 + c}{\pm\sqrt{a^2 + b^2}},$$

we have in this case $d = \dfrac{1 + 4 + 1}{-\sqrt{1 + 1}} = -\dfrac{6}{\sqrt{2}} = -3\sqrt{2} = -4.24+.$

Draw the figure and note that the result thus obtained by the formula is correct in sign as well as numerically.

2. Find the distance from the line $3x + 4y + 1 = 0$ to each of the points $(-1, -3)$, $(2, 5)$, and $(0, -4)$.

3. Find the distance from the line $y - 2x = 0$ to each of the points $(3, -2)$, $(1, 4)$, and $(-2, 2)$.

4. Find the equation of the locus of a point that moves so as to be equally distant from the lines

$$3x - 4y + 5 = 0 \quad (1)$$

and

$$5x + 12y - 6 = 0. \quad (2)$$

Solution. Let $P \equiv (x, y)$ be any point on the locus, and d_1 and d_2 the distances from the given lines to the point P (Fig. 74).

Then

$$d_1 = \frac{3x - 4y + 5}{-5}$$

and

$$d_2 = \frac{5x + 12y - 6}{13}.$$

Since the point P is equally distant from the lines (1) and (2), we must have either

$$d_1 = d_2$$

or

$$d_1 = -d_2;$$

that is, either

$$\frac{3x - 4y + 5}{-5} = \frac{5x + 12y - 6}{13} \tag{A}$$

or

$$\frac{3x - 4y + 5}{-5} = -\frac{5x + 12y - 6}{13}. \tag{B}$$

(A) contains all points equally distant from (1) and (2) for which the distances are *both positive* or *both negative*; (B) contains all points equally distant from (1) and (2) for which one of the distances is positive and

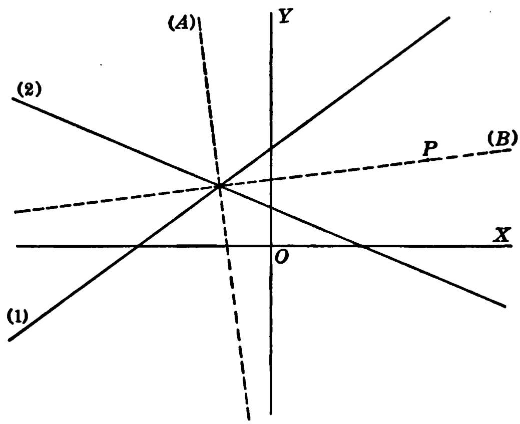


FIG. 74

the other negative. These loci are of course the bisectors of the angles formed by the lines (1) and (2), and (A) is the one passing through the angle *containing the origin.*

The simplified forms of the equations are

$$64x + 8y + 35 = 0 \tag{A}$$

and

$$14x - 112y + 95 = 0. \tag{B}$$

5. Find the equations of the bisectors of the angles formed by the lines $3x + 4y = 6$ and $3x - 4y = 10$; by the lines $x - 2y + 3 = 0$ and $2x - y - 7 = 0$.

6. Find the equations of the bisectors of the angles formed by the lines $3x - y = 0$ and $x + 3y - 4 = 0$.

7. Find the equations of the interior bisectors of the angles of the triangle formed by the lines $5x - 12y = 0$, $5x + 12y + 20 = 0$, and $12x - 5y - 30 = 0$. Show that they meet in a point.

8. Find the equations of the interior bisectors of the angles of the triangle formed by the lines $4x - 3y = 5$, $5x - 12y = 10$, and $4x + 3y = 12$. Show that they meet in a point.

9. Answer the same question for the triangle formed by the lines $y = 4$, $y = 2x$, and $y = -2x$.

MISCELLANEOUS PROBLEMS ON THE STRAIGHT LINE

1. Find the equations of the medians of the triangle whose vertices are the points $(2, 3)$, $(4, -5)$, and $(-2, -1)$. Show that they meet in a point.

2. Find the equations of the perpendicular bisectors of the sides of the same triangle, and show that they meet in a point.

3. Find the equations of the altitudes of the above triangle, and show that they meet in a point.

4. Show that the three points of intersection obtained in Problems 1–3 lie on a straight line.

5. Find the center and radius of the circumcircle of the triangle whose vertices are the points $(0, 5)$, $(7, -3)$, and $(-2, 2)$.

6. Find the area of the triangle whose vertices are the points $(0, 0)$, $(-2, 3)$, and $(-4, -1)$.

7. Prove that the area of the triangle whose vertices are the points $(0, 0)$, (x_1, y_1), and (x_2, y_2) is $\frac{1}{2}(x_1y_2 - x_2y_1)$.

8. Find the center and radius of the inscribed circle of the triangle whose sides are the lines $3x + 4y - 8 = 0$, $4x + 3y + 6 = 0$, and $5x - 12y - 13 = 0$.

9. Find the center and radius of each of the escribed circles[1] of the triangle in Problem 8.

10. Prove that the medians of any triangle meet in a point. (Take the origin at one vertex and let the X-axis coincide with one side; that is, let the vertices have the coördinates $(0, 0)$, $(a, 0)$, and (b, c).)

[1] An escribed circle of a triangle is a circle which is tangent to one side of the triangle and to the other two sides produced.

11. Prove that the perpendicular bisectors of the sides of any triangle meet in a point.

12. Prove that the altitudes of any triangle meet in a point.

13. Prove that the three points of intersection found in Problems 10, 11, and 12 lie on a straight line.

14. Show that the locus of the equation

$$(ax + by + c)(a'x + b'y + c') = 0$$

consists of the two lines $ax + by + c = 0$ and $a'x + b'y + c' = 0$.

15. Prove that the locus of the equation $ax^2 + bxy + cy^2 = 0$ is a pair of intersecting lines if $b^2 - 4ac > 0$, that it is one straight line if $b^2 - 4ac = 0$, and that it contains no real point except the origin if $b^2 - 4ac < 0$.

16. Prove that the two straight lines $bx^2 - cxy + ay^2 = 0$ are respectively perpendicular to the lines $ax^2 + cxy + by^2 = 0$.

17. Taking the vertices of a triangle as (x_1, y_1), (x_2, y_2), and (x_3, y_3), find the equations of the perpendicular bisectors of the sides. Show that they meet in a point.

18. Prove that the locus of the equation

$$ax + by + c + k(a'x + b'y + c') = 0$$

is a straight line through the points of intersection of the lines $ax + by + c = 0$ and $a'x + b'y + c' = 0$.

19. Prove that the area of the triangle whose vertices are the points (x_1, y_1), (x_2, y_2), and (x_3, y_3) is $\frac{1}{2}(x_1y_2 - x_2y_1 + x_2y_3 - x_3y_2 + x_3y_1 - x_1y_3)$, that is,

$$\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}.$$

The Circle

84. Several of the locus problems in the preceding chapter (for example, Exs. 11, 12, p. 95) led to circles as their result. Indeed, it is easy to determine the equation of any circle in terms of its radius and the coördinates of the center.

Let $C \equiv (\alpha, \beta)$ be the center and r the radius (Fig. 75). If $P \equiv (x, y)$ is any point on the circle, then

$$CP = r.$$

But $CP = \sqrt{(x-\alpha)^2 + (y-\beta)^2}$.

Therefore

$$(x-\alpha)^2 + (y-\beta)^2 = r^2, \quad (1)$$

which is accordingly the equation of the circle.

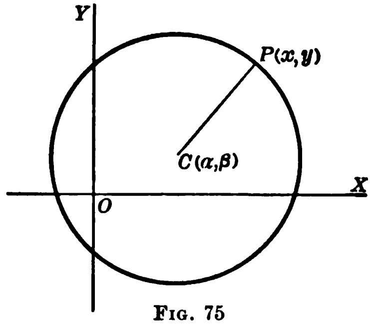


FIG. 75

COROLLARY. The equation of the circle whose center is the origin and whose radius is r is

$$x^2 + y^2 = r^2. \quad (2)$$

EXERCISES

1. Draw the graphs of the following equations: $x^2 + y^2 = 1$, $x^2 + y^2 = 2$, $x^2 + y^2 = \frac{1}{2}$, $x^2 + y^2 = 3.24$.

2. Write the equation of the circle whose center is

(a) (4, 6) and whose radius is 5;
(b) $(-1, -2)$ and whose radius is 3;
(c) (2, 0) and whose radius is 2;
(d) $(a, 0)$ and whose radius is a.

3. Show, by reducing to the form (1), that the locus of each of the following equations is a circle, and construct it.

(a) $x^2 + y^2 - 4x + 2y + 1 = 0$.

Solution. To get the standard form (1), complete the square of the terms in x and also of those in y, thus:

$$(x^2 - 4x + 4) + (y^2 + 2y + 1) = 4.$$

That is,
$$(x-2)^2 + (y+1)^2 = 4.$$

This is in the form (1), with $\alpha = 2$, $\beta = -1$, $r^2 = 4$; hence the graph is a circle whose center is $(2, -1)$ and whose radius is 2. The student may now draw the figure.

(b) $x^2 + y^2 - x - 4y = 2$.
(c) $x^2 + y^2 + 16x - 12y + 84 = 0$.
(d) $x^2 + y^2 + 4x + 8y - 41 = 0$.
(e) $x^2 + y^2 - x - y = 0$.
(f) $x^2 + y^2 = 3(x + 3)$.
(g) $3x^2 + 3y^2 - 5x + 24y = 0$.
(h) $x^2 + y^2 + 16y + 36 = 0$.

4. Find the equation of the circle whose center is $(3, 0)$ and which passes through the origin.

5. Find the equation of the circle whose center is $(2, -1)$ and which passes through the point $(-1, -5)$.

6. Find the equation of the circle whose center is $(3, -2)$ and which is tangent to the Y-axis.

7. Find the equation of the circle which has the line joining the points $(3, 4)$ and $(-1, -2)$ as diameter.

8. Find the equation of the circle whose center is on the X-axis and which is tangent to the lines $y = 3$ and $x = 1$. (Two solutions.)

9. Find the equation of the circle whose center is on the Y-axis and which is tangent to the lines $y = 2x - 1$ and $y = -2x$; of the circle whose center is on the X-axis and which is tangent to the same two lines.

10. Find the equation of the circle whose center is on the line $y = 1$ and which is tangent to each of the lines $3x - y = 6$ and $x - 3y = 3$.

11. Find the equation of the circle that is tangent to the three lines $x = 0$, $y = 0$, and $x = 3$.

12. Find the equation of the circle that is tangent to the three lines $x = 5$, $y = x$, and $y = -x$. (Four solutions.)

13. Find the equation of the circle that is tangent to the three lines $3x - 4y - 5 = 0$, $4x + 3y - 8 = 0$, and $4x - 3y + 12 = 0$.

14. Find the equation of the circle whose center is on the Y-axis and which passes through the points $(3, 4)$ and $(2, -1)$.

15. Find the equation of the circle which passes through the three points $(4, -1)$, $(-2, -3)$, and $(-1, 3)$.

16. Find the equation of the circle which passes through the three points $(0, 5)$, $(3, 0)$, and $(-2, -4)$.

17. Find the equation of the circle which passes through the points $(-1, 9)$ and $(6, 8)$ and is tangent to the X-axis.

Ans. $x^2 + y^2 - 4x - 10y + 4 = 0$
or $x^2 + y^2 - 244x - 1690y + 14{,}884 = 0$.

18. Find the equation of the circle which passes through the points (8, 8) and (1, 7) and is tangent to the line $3x + 4y = 6$.

Ans. $x^2 + y^2 - 10x - 8y + 16 = 0$
or $x^2 + y^2 - 2x - 64y + 400 = 0$.

19. Find the equation of the circle which passes through the point (2, 4) and is tangent to the X- and Y-axes. (Two solutions.)

Prove that each of the following loci (Exs. 20–22) is a circle, by showing that its equation assumes the form (1) (p. 111). To this end, choose the X- and Y-axes in a convenient position with reference to the given points or lines.

20. A point moves so that the sum of the squares of its distances from two fixed points is constant.

21. A point moves so that the ratio of its distances from two fixed points is a constant (not equal to 1).

22. A point moves so that the square of its distance from a fixed point, divided by its distance from a fixed line, is constant.

23. Find the equation of the circle passing through the mid-points of the sides of the triangle whose vertices are (4, 0), (− 2, 0), and (0, 6).

24. Prove that the circle of Ex. 23 passes through the feet of the altitudes of the triangle, and also through the points halfway between the vertices and the orthocenter of the triangle. This circle is called the Nine Points Circle of the triangle.

25. Do as in Exs. 23, 24 for the triangle whose vertices are $(a, 0)$, $(b, 0)$, and $(0, c)$. *Ans.* The equation of the Nine Points Circle is

$$x^2 + y^2 - \frac{a+b}{2}x + \frac{ab - c^2}{2c}y = 0.$$

CHAPTER VII

THE PARABOLA, ELLIPSE, AND HYPERBOLA

85. The parabola. A particularly important locus is that of a point which moves so as to be equally distant from a fixed point and from a fixed line. The student should construct carefully such a locus, taking the fixed point and the fixed line quite at random, and locating a sufficient number of points on the locus to make the shape of the curve clear (cf. Ex. 14, p. 95). This locus is called a *parabola*, and thus we have for the first time a definition of that word, which we have often used heretofore merely for the sake of giving a name to a curve of a certain shape. We have now the definition:

A parabola is the locus of a point whose distance from a fixed point is always equal to its distance from a fixed line. The fixed point is called the *focus*, and the fixed line the *directrix*, of the parabola.

86. Equation of the parabola. The loci of Exs. 14–17, pp. 95, 96, were parabolas, and accordingly the equations obtained were equations of parabolas. In order to get the simplest possible form of equation, proceed as follows:

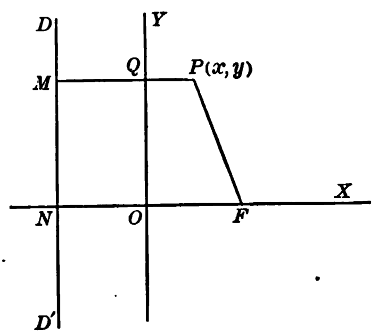


FIG. 76

Let F be the focus and DD' the directrix. Draw the perpendicular from F to DD', and choose this line as the X-axis, with the direction from DD' to F as the positive direction. Choose as origin the point midway between F and DD', and represent the distance OF by a. Then the coördinates of F are $(a, 0)$, and the equation of DD' is $x = -a$. Now if $P \equiv (x, y)$

is any point on the locus, the two distances MP and FP must be equal, by the definition of the parabola.

But $$MP = x + a$$
and $$FP = \sqrt{(x-a)^2 + y^2}.$$
Therefore $$\sqrt{(x-a)^2 + y^2} = x + a \qquad (1)$$
Simplifying, we get $$\boldsymbol{y^2 = 4\,ax}. \qquad (2)$$

This equation must then be satisfied by the coördinates of all points on the parabola; and, conversely, all points whose coördinates satisfy (2) must lie on the parabola, because we can reason back from (2) to (1). The possible double sign in (1) does not make any difference here, as $x + a$ cannot be negative.

Equation (2) is therefore the equation of the parabola, and it may be considered as a *standard form* for the equation of the curve. By analyzing the equation several important properties of the curve may easily be obtained.

87. (1) Solving (2) for y, we have

$$y = \pm\sqrt{4\,ax}.$$

Hence, for any positive value of x, y has *two* corresponding values, which are equal numerically but of opposite sign. This means that the curve is symmetrical with respect to the X-axis. The line through the focus, perpendicular to the directrix, of a parabola is accordingly called the *axis of symmetry*, or simply the *axis*, of the curve. The point where it intersects the parabola is called the *vertex* of the curve. Here the vertex is the origin.

(2) The form of the equation $y^2 = 4\,ax$ shows that x and a must have the same sign, since y^2 cannot be negative; that is, x must be *positive*, since we have taken a as positive. Hence the curve lies entirely on the positive side of the Y-axis.

(3) As x increases, the positive value of y increases also, but less rapidly (since y is proportional to the *square root* of x), so that for very large values of x a small change in x will produce scarcely any change at all in y. This means that the curve becomes more and more nearly horizontal as it recedes from the vertex.

(4) The chord of the parabola drawn through the focus perpendicular to the axis is called the *latus rectum*. The student may show that the coördinates of B (Fig. 77) are $(a, 2a)$ and that those of C are $(a, -2a)$, and hence that the length of the latus rectum is $4a$.

Summarizing, the equation $y^2 = 4ax$ has for its locus the parabola with focus at $(a, 0)$, directrix $x = -a$, axis the X-axis, vertex the origin, and latus rectum $4a$.

If we take a as a negative quantity, the parabola will be on the *negative* side of the Y-axis, and the other inferences that have been drawn will require corresponding modifications. Also, if we should choose the line through the focus, perpendicular to the directrix (that is, the axis of the parabola), as Y-axis instead of as X-axis, the effect would be merely to interchange x and y in the equation (2), giving

$$x^2 = 4ay. \qquad (3)$$

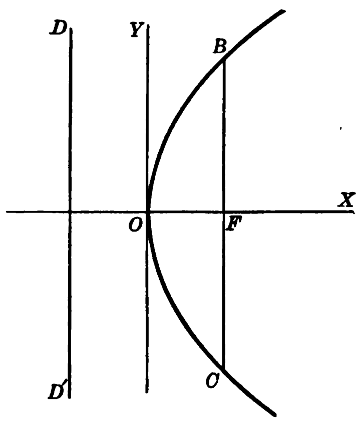


FIG. 77

The focus is now the point $(0, a)$, the vertex is the origin, and the curve is concave upward if a is positive, concave downward if a is negative. Draw the figure for this choice of coördinate axes, taking the Y-axis vertical as usual. What is the equation of the directrix?

EXERCISES

Draw the loci of the following equations, locate vertex, focus, directrix, and axis of symmetry, and find the length of the latus rectum:

1. $y^2 = 8x$.	5. $y^2 = \frac{4}{3}x$.	9. $x^2 = -5y$.
2. $y^2 = 10x$.	6. $y^2 = -4x$.	10. $y = 4x^2$.
3. $y^2 = x$.	7. $y^2 = -16x$.	11. $x^2 + 3y = 0$.
4. $y^2 = \frac{1}{2}x$.	8. $x^2 = 4y$.	12. $y^2 + \frac{1}{2}x = 0$.

***88. Generalized standard equation of the parabola.** We often have to deal with problems in which the X- and Y-axes cannot be chosen at pleasure but are already prescribed in a less favorable position than that which we chose in deriving the standard equation (2), § 86. Exs. 14–17, pp. 95, 96, are examples of such problems. In problems of this kind the equation of the curve will not take so simple a form as (2) or (3), §§ 86 and 87. The simplest generalization, and the only one which we shall here consider, is the case where the axis of the parabola is parallel to either the X- or the Y-axis and the vertex and the latus rectum are given. In this case the equation of the curve can be found by using a certain fundamental property of the parabola, as follows:

If $P \equiv (x, y)$ is any point on the parabola indicated in Fig. 78, we have seen that the coördinates of P must satisfy the equation

$$y^2 = 4\,ax.$$

But $y = MP$ and $x = VM$, so that the equation $y^2 = 4\,ax$ is the same as

$$MP^2 = 4\,a \cdot VM.$$

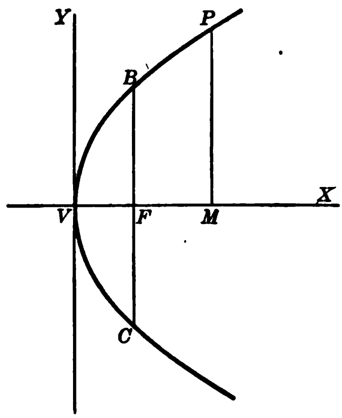


FIG. 78

This relation must then hold true for all points P on the parabola whose vertex is V and whose latus rectum is $4\,a$. It accordingly states *an intrinsic property of the curve,* which, expressed in words, is the following

THEOREM. *If from any point on a parabola a perpendicular is drawn to the axis of the curve, then the square of this perpendicular* (MP^2) *is equal to the product of the latus rectum* ($4\,a$) *by the distance from the vertex to the foot of the perpendicular* (VM).

This theorem is now stated in such a way as to be independent of the position of the coördinate axes. We can use it to write down at once the equation of a parabola whose axis is parallel to either the X- or the Y-axis. Let the vertex be

$V \equiv (\alpha, \beta)$ (Fig. 79), and let the latus rectum be $4a$, the axis being parallel to the X-axis. Then if $P \equiv (x, y)$ is any point on the curve, we have, by the above theorem,

$$MP^2 = 4a \cdot VM.$$

But $MP = SP - SM$
$= SP - QV$
$= y - \beta,$

and $VM = RM - RV$
$= OS - OQ$
$= x - \alpha.$

Therefore

$$(y - \beta)^2 = 4a(x - \alpha). \quad (4)$$

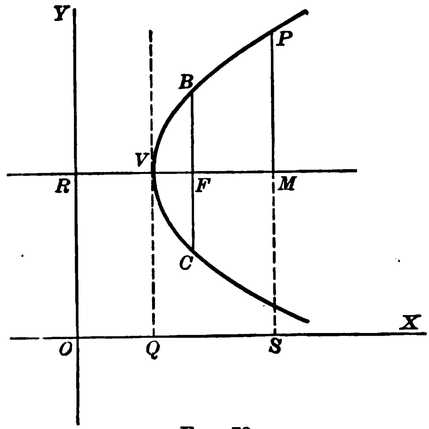


FIG. 79

Since this equation is true for all points (x, y) on the parabola, and for no other points, it is the equation of the curve.

If the axis of the curve had been parallel to the Y-axis instead of to the X-axis, the equation would have been

$$(x - \alpha)^2 = 4a(y - \beta). \quad (5)$$

The student should draw the figure and give the proof for this case also.

EXERCISE

Give the coördinates of F, B, and C, and the equation of the directrix, both for Fig. 79 and for the case where the axis of the parabola is parallel to the Y-axis.

***89.** SUMMARY. The equation $(y - \beta)^2 = 4a(x - \alpha)$ represents a parabola with vertex at the point (α, β), but otherwise the same as the locus of the equation $y^2 = 4ax$. Its locus can be thought of, indeed, as the same curve as $y^2 = 4ax$, merely moved along so that its vertex takes the position (α, β), the axis of the curve remaining horizontal. Likewise, the equation $(x - \alpha)^2 = 4a(y - \beta)$ represents the same parabola as $x^2 = 4ay$, moved so that its vertex takes the position (α, β), the axis of the curve remaining vertical. A movement of this sort is called a *translation*.

***90.** By using these results we can also recognize and draw the graph of any equation of a parabola whose axis is parallel to the X-axis or to the Y-axis.

Example. Draw the graph of $y^2 + 2x - 4y = 6$. This equation can be reduced to the form (4),
$$(y - \beta)^2 = 4a(x - \alpha),$$
by completing the square of the y-terms.

Thus,
$$y^2 - 4y = -2x + 6.$$
Adding 4 to complete the square,
$$y^2 - 4y + 4 = -2x + 10;$$
that is,
$$(y - 2)^2 = -2(x - 5).$$
This is now in the form (4) with $\alpha = 5$, $\beta = 2$, $4a = -2$ (that is, $a = -\frac{1}{2}$). Hence the graph is the parabola whose vertex is the point (5, 2), whose axis is parallel to the X-axis, and whose latus rectum is 2. The focus is to the *left* of the vertex, because a is negative. Hence the coördinates of the focus are $(4\frac{1}{2}, 2)$, and the equation of the directrix is $x = 5\frac{1}{2}$. With this information it is easy to draw the graph without computing the coördinates of any points on the curve. As a check, however, it is always well to determine the intercepts on the X- and Y-axes. Thus, when $y = 0$, $2x = 6$, $x = 3$; hence the point (3, 0) is on the curve. When $x = 0$, $y^2 - 4y - 6 = 0$, $y = 2 \pm \sqrt{10} = 5.16+$ or $-1.16+$. Hence the points $(0, 5.16+)$ and $(0, -1.16+)$ are on the curve. These facts agree with Fig. 80.

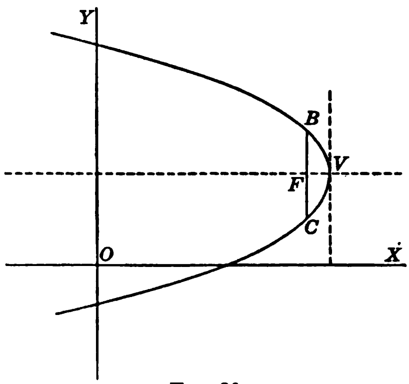


FIG. 80

EXERCISES

Draw the graphs of the following equations and locate vertex, focus, and directrix. Check by finding the intercepts of the curve on the X- and Y-axes.

1. $y^2 - 2x - 2y = 1.$
2. $y^2 - 5x - 8y + 1 = 0.$
3. $y^2 - x - y = 0.$
4. $4y^2 - 6x - y = 5.$
5. $3y^2 + 4x - 5y + 2 = 0.$
6. $2y^2 + \frac{1}{2}x - y + \frac{3}{4} = 0.$
7. $x^2 - 8x + 3y = 0.$
8. $x^2 - x + 2y = 3.$
9. $4x^2 - 9x + y - 1 = 0.$
10. $10x^2 - 15x + 8y = 20.$

91. Construction of parabola. (1) *By points.* Since any point on the parabola is equally distant from the focus and from the directrix, we can locate any number of points on the curve as follows:

Let F (Fig. 81) be the focus and DD' the directrix. Draw the axis AFX, and through Q, any point on the axis, draw RS perpendicular to AX. From F as center, with radius AQ, describe an arc meeting RS in the points P and P'. These will both be points on the parabola whose focus is F and whose directrix is DD'. (Why?)

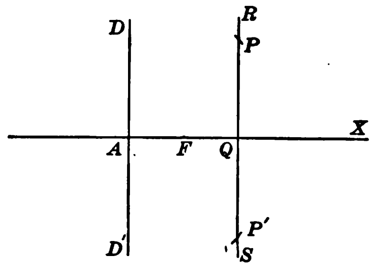


FIG. 81

(2) *By continuous motion.* Place a right triangle JKH (Fig. 82) with KH along the axis AX and with KJ along DD'. One end of a string of length KH is fastened at H, the other end at F. If now a pencil point P be pressed against the string, keeping it taut as the triangle JKH is slid along the directrix, then P will trace an arc of a parabola. Prove this.

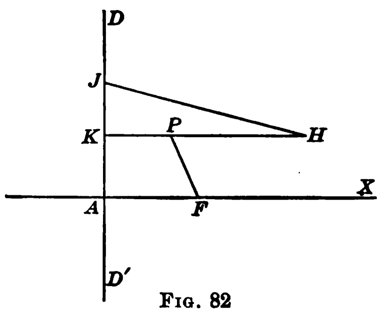


FIG. 82

PROBLEMS ON THE PARABOLA

1. The expression $y^2 - 4ax$, which equals zero for all points (x, y) on the parabola whose equation is $y^2 = 4ax$, is positive for all points outside the curve and negative for all points inside.

2. Every point outside the parabola is nearer to the directrix than to the focus; every point inside the parabola is nearer to the focus than to the directrix.

3. If PP' is any chord of a parabola passing through the focus, then the circle on PP' as diameter is tangent to the directrix.

4. If FP is the line segment joining the focus to any point on the parabola $y^2 = 4\,ax$, then the circle on FP as diameter is tangent to the Y-axis.

5. Given the directrix of a parabola, and two points on the curve, to find the focus.

6. Given the focus of a parabola and two points on the curve, to find the directrix.

7. The straight line passing through the mid-points of any two parallel chords of a parabola is parallel to the axis of the curve and bisects all other chords that are parallel to the original ones.

8. Given a parabola, to find its axis, focus, and directrix.

9. The vertex of a parabola is O, and P is any other point on the curve. Through P two lines are drawn, one perpendicular to the axis and the other perpendicular to OP. These lines meet the axis in Q and R respectively. Prove that QR is equal to the latus rectum.

10. Two perpendicular chords are drawn through the vertex of a parabola. Show that the line joining their other intersections with the parabola meets the axis in a fixed point.

11. Prove that the following is a correct construction for a parabola: A being the vertex and F the focus, produce AF to B, making $FB = AF$, and draw the circle with B as center and BA as radius. At Q, any point on the axis, erect the perpendicular QR to AB (R being on the circle), and lay off QP (and QP') equal to AR. Then P and P' are points on the parabola. In this way any number of points can be found.

12. Given A and F, the vertex and focus respectively, draw FA and produce it to C, so that $AC = 4\,FA$. Let Q be any point on the axis FA, and draw the circle having CQ as diameter. Draw the chord $RAR' \perp CQ$, and the tangent QP. The line $RP \parallel AQ$ meets QP in P, which is a point of the parabola. Prove this.[1]

The Ellipse

92. Definition of the ellipse. Another important locus is that of a point which moves so as to be always half as far from a fixed point as from a fixed line. The student should construct this locus carefully, taking a point and a line entirely at random

[1] The constructions of Exs. 11 and 12 are of Arabian origin. They are found in a work written by an Arabian mathematician named Abû'l Wâfâ, who lived in the tenth century.

and locating enough points on the locus to make the shape of the curve evident (cf. Exs. 20, 21, p. 96). This locus is called an *ellipse.* The locus would also be an ellipse if we changed the word "half" in the first sentence of this paragraph to "one third," "two thirds," or any other positive number less than one. The complete definition of an ellipse is accordingly:

An ellipse is the locus of a point whose distance from a fixed point is to its distance from a fixed line in a constant ratio less than one. The fixed point is called the *focus,* the fixed line the *directrix,* and the constant ratio the *eccentricity.*

The student should now construct carefully several ellipses, using different values of the eccentricity, — for instance, $\frac{1}{3}$, $\frac{2}{3}$, $\frac{3}{4}$, $\frac{1}{5}$, $\frac{4}{5}$, $\frac{9}{10}$. Notice how the shape of the curve is affected by increasing or diminishing the eccentricity.

93. Equation of the ellipse. The simplest form of the equation of an ellipse can be obtained as follows: Let F be the focus, DD' the directrix, and e the eccentricity, a positive number less than one. Choose as X-axis the line through the focus perpendicular to the directrix, with the direction from F to DD' as the positive direction. Two points on the curve can be located at once, namely, the points A and A' where the curve meets the X-axis; for

$$\frac{FA}{AB} = e \tag{1}$$

and

$$\frac{A'F}{A'B} = e, \tag{2}$$

so that A and A' can be found by a simple construction of elementary geometry. Take now as origin the mid-point of the segment $A'A$, and represent the distance OA by a. Then $A'O$ also equals a. In order to get the equation of the ellipse in the simplest possible form, we shall need to obtain the lengths OF and OB in terms of a and e. To accomplish this, rewrite the equations (1) and (2) above:

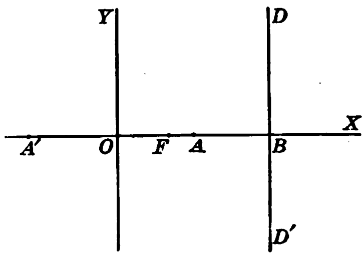


FIG. 83

$$FA = e \cdot AB, \tag{1}$$

$$A'F = e \cdot A'B. \tag{2}$$

Adding, we get $A'F + FA = e(A'B + AB)$. (3)

But $A'F + FA = A'A = 2a$,

and
$$A'B + AB = (A'O + OB) + (OB - OA)$$
$$= 2\,OB,$$

since $A'O = OA$.

Therefore (3) gives $2a = e \cdot 2\,OB$;

that is,
$$\boldsymbol{OB = \frac{a}{e}}. \tag{4}$$

In order to get the length of OF, subtract (1) from (2):
$$A'F - FA = e(A'B - AB) \tag{5}$$
$$= e \cdot A'A = 2\,ae.$$

But
$$A'F - FA = (A'O + OF) - (OA - OF)$$
$$= 2\,OF,$$

since $A'O = OA$.

Therefore (5) gives $2\,OF = 2\,ae$;

that is,
$$\boldsymbol{OF = ae}. \tag{6}$$

The important results of equations (4) and (6) may be stated thus: the coördinates of F are $(ae, 0)$, and the equation of DD' is $x = \frac{a}{e}$.

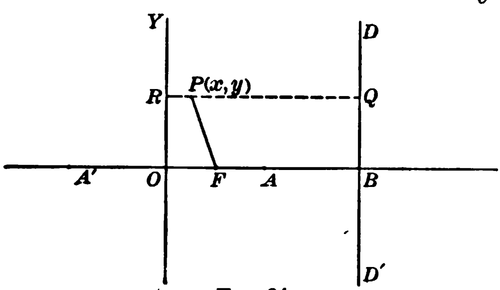


FIG. 84

Now, to get the equation of the ellipse, let $P \equiv (x, y)$ be any point on the curve; then, by the definition of the ellipse,
$$\frac{PF}{PQ} = e,$$

or $PF = e \cdot PQ$. (7)

But
$$PF = \sqrt{(x - ae)^2 + y^2}$$

and
$$PQ = RQ - RP = \frac{a}{e} - x.$$

Therefore
$$\sqrt{(x - ae)^2 + y^2} = e\left(\frac{a}{e} - x\right) = a - ex, \tag{8}$$

or
$$(x - ae)^2 + y^2 = (a - ex)^2.$$

Multiplying out and rearranging terms,

$$x^2(1-e^2)+y^2=a^2(1-e^2). \tag{9}$$

Equation (9) is the equation of the ellipse, because it is the equation which must be satisfied by the coördinates (x, y) of all points on the curve; and it is not satisfied by the coördinates of any other points, because, if (9) is true, we can reason backwards through (8) to (7), the possible change of sign in (8) affecting only the direction of PQ, which here makes no difference, because e is the ratio of the numerical lengths of PF and PQ.

94. It is usual to make, in equation (9), the abbreviation $b^2=a^2(1-e^2)$, which is allowable because $e<1$ and hence $a^2(1-e^2)$ is necessarily positive. Then (9) becomes

$$\frac{b^2}{a^2}x^2+y^2=b^2;$$

that is,

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1. \tag{10}$$

This is the simplest form of the equation of an ellipse, and it may be considered as a *standard form* for the equation of the curve. By analyzing the equation, several important properties of the curve may be obtained.

95. (1) Solving (10) for y as an explicit function of x, we obtain $y=\pm\frac{b}{a}\sqrt{a^2-x^2}$. This shows that x^2 cannot be $>a^2$; for if it were, y would be complex, thus giving no point on the curve. This means that no point of the curve lies to the right of $x=a$ (the point A) or to the left of $x=-a$ (the point A'). For any value of x between $-a$ and $+a$, y has two values, numerically equal but opposite in sign, so that the curve is symmetrical with respect to the X-axis.

(2) Solving (10) for x as an explicit function of y, and reasoning as in the preceding paragraph, we find that the ellipse is symmetrical to the Y-axis, and that the points $B\equiv(0, b)$ and $B'\equiv(0, -b)$ are respectively the highest and the lowest point on the curve. The segment $B'B$ is called the *minor axis* of the ellipse;

$A'A$ is called the *major axis*. The end-points of the axes, A, A', B, and B', are called the *vertices* of the ellipse. The intersection of the axes, O, is called the *center* and is the mid-point of any chord drawn through it.[1]

(3) The relation among the three quantities a, b, and e (namely, $b^2 = a^2(1-e^2) = a^2 - a^2e^2$) may be easily remembered by noting that in a right triangle with b and ae as the two legs, a will be the hypotenuse. Thus, in Fig. 85, the triangle BOF has $OF = ae$, $OB = b$; hence $BF = a$. This fact enables us to find the focus of an ellipse when the major and minor axes are given.

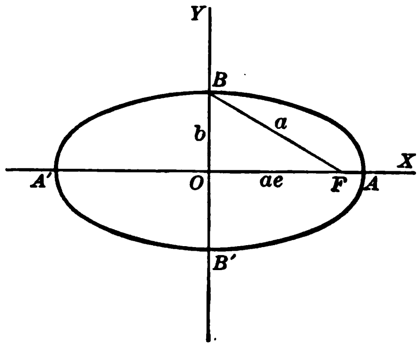


FIG. 85

(4) As in the parabola, the chord through the focus, perpendicular to the major axis, is called the latus rectum. The student may show that the coördinates of its end-points are

$$\left(ae, \frac{b^2}{a}\right) \text{ and } \left(ae, -\frac{b^2}{a}\right), \text{ so that its length is } \frac{2\,b^2}{a}.$$

96. Summarizing, the equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ has for its locus the ellipse with major axis $2\,a$, minor axis $2\,b$, center at the origin, vertices $(\pm\, a, 0)$ and $(0, \pm\, b)$, focus $(ae, 0)$, and directrix $x = \frac{a}{e}$, where the value of e may be obtained from the fact that $a^2e^2 = a^2 - b^2$. Thus, the equation $\frac{x^2}{5} + y^2 = 1$ has for its locus the ellipse with center at the origin, major axis $2\sqrt{5}$, minor axis 2, vertices $(\pm\sqrt{5}, 0)$ and $(0, \pm 1)$, focus $(2, 0)$, and directrix $x = \frac{5}{2}$ (Fig. 86).

[1] Because if (x_1, y_1) is a point satisfying the equation of the ellipse, $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, then $(-x_1, -y_1)$ will also satisfy the equation; and the origin is the mid-point of the segment joining the points (x_1, y_1) and $(-x_1, -y_1)$.

In deriving the equation of the ellipse, if we had chosen the line through the focus, perpendicular to the directrix, as Y-axis instead of as X-axis, the effect would have been merely to interchange x and y in the equation (10) (§ 94), giving

$$\frac{y^2}{a^2}+\frac{x^2}{b^2}=1. \quad (11)$$

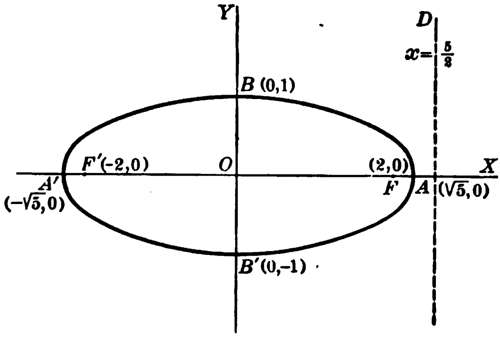


FIG. 86

Thus, the equation $\frac{y^2}{3}+\frac{x^2}{2}=1$ has for its graph the ellipse with center at the origin, major axis $2\sqrt{3}$, minor axis $2\sqrt{2}$, vertices $(0, \pm\sqrt{3})$ and $(\pm\sqrt{2}, 0)$, focus $(0, 1)$, and directrix $y=3$. It is the same curve as the ellipse $\frac{x^2}{3}+\frac{y^2}{2}=1$, rotated about the origin through an angle of 90°. Draw the figure.

EXERCISES

Draw the graphs of the following equations, locating focus and directrix and finding the eccentricity:

1. $\frac{x^2}{9}+\frac{y^2}{4}=1.$
2. $\frac{x^2}{4}+y^2=1.$
3. $\frac{x^2}{25}+\frac{y^2}{16}=1.$
4. $\frac{x^2}{4}+\frac{y^2}{2}=1.$
5. $\frac{x^2}{5}+y^2=1.$
6. $x^2+\frac{y^2}{4}=1.$
7. $\frac{x^2}{3}+\frac{y^2}{7}=1.$
8. $7x^2+3y^2=21.$
9. $x^2+\frac{1}{2}y^2=\frac{1}{3}.$
10. $2x^2+3y^2=1.$
11. $4x^2+5y^2=6.$

12. Prove that the point $(-ae, 0)$ and the line $x=-\frac{a}{e}$ are also a focus and directrix of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$

13. Find the equation of the ellipse whose center is the origin and whose foci are on the X-axis, if $a=10$ and $e=\frac{3}{5}$.

***97. Generalized standard equation of the ellipse.** As in the case of the parabola, the simple standard equation of the ellipse gives a geometric property common to all points of the curve, and this property will enable us to obtain a more general form for the equation of the curve. Thus, $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is equivalent (see Fig. 87) to $\frac{NP^2}{a^2}+\frac{MP^2}{b^2}=1$, this relation being expressed in words by the following

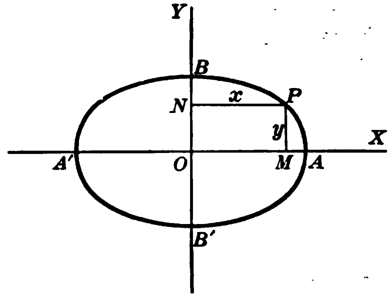


Fig. 87

THEOREM. *If from any point on an ellipse perpendiculars are drawn to the major and minor axes, the square of the perpendicular upon the minor axis* (NP^2), *divided by the square of the semi-major axis* (a^2), *plus the square of the perpendicular upon the major axis* (MP^2), *divided by the square of the semi-minor axis* (b^2), *equals unity.*

Using this theorem, we can write down the equation of an ellipse whose axes are parallel to the X- and Y-axes, and whose center is any point (α, β). Let the major axis be parallel to the X-axis and of length $2a$, the minor axis having the length $2b$. Then, by the theorem just stated,

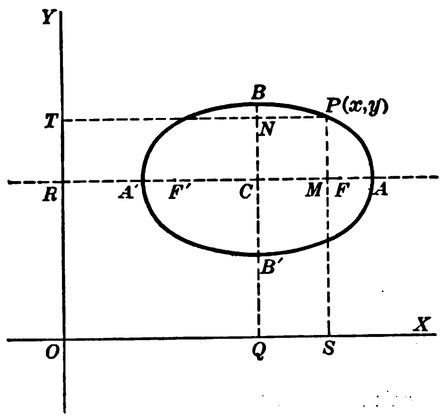


Fig. 88

$$\frac{NP^2}{a^2}+\frac{MP^2}{b^2}=1.$$

But
$$\begin{aligned} NP &= TP - TN \\ &= OS - OQ \\ &= x - \alpha, \end{aligned}$$

and
$$\begin{aligned} MP &= SP - SM \\ &= y - \beta. \end{aligned}$$

Therefore the equation of the ellipse is

$$\frac{(x-\alpha)^2}{a^2}+\frac{(y-\beta)^2}{b^2}=1. \tag{12}$$

If the major axis had been parallel to the Y-axis, we should have had

$$\frac{(x-\alpha)^2}{b^2}+\frac{(y-\beta)^2}{a^2}=1. \tag{13}$$

The figure should be drawn and the proof given for this case also.

EXERCISE

Give the coördinates of F, F', A, A', B, B', and the equation of each directrix (both for equation (12) and for equation (13)).

***98.** Summary. The equation $\frac{(x-\alpha)^2}{a^2}+\frac{(y-\beta)^2}{b^2}=1$ represents an ellipse with center at (α, β), but otherwise the same curve as the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. It can be thought of, indeed, as the same curve, merely translated from the position in which the origin is the center to the position in which the point (α, β) is the center.

By using these results we can recognize and hence draw the graph of any equation that is reducible to either of the forms (12) or (13).

Example. Draw the graph of the equation $x^2+4y^2-6x+4y-6=0$. We see that by completing the square of the x-terms and of the y-terms it will be possible to reduce this to the form (12) or (13). Hence we write it first in the form

$$(x^2-6x)+4(y^2+y)=6.$$

To complete the square we must add 9 inside the first parenthesis and $\frac{1}{4}$ inside the second (which amounts to adding $9+1$ to the left member of the equation):

$$(x^2-6x+9)+4(y^2+y+\tfrac{1}{4})=6+9+1=16;$$

that is,

$$(x-3)^2+4(y+\tfrac{1}{2})^2=16,$$

or

$$\frac{(x-3)^2}{16}+\frac{(y+\frac{1}{2})^2}{4}=1,$$

which is in the form (12) with $\alpha=3$, $\beta=-\frac{1}{2}$, $a=4$, $b=2$. Hence the graph is the ellipse whose center is the point $(3, -\frac{1}{2})$, whose major axis

is 8, and whose minor axis is 4. The vertices are accordingly the points $(7, -\frac{1}{2})$, $(-1, -\frac{1}{2})$, $(3, 1\frac{1}{2})$, and $(3, -2\frac{1}{2})$. To find the focus,

$$a^2e^2 = a^2 - b^2 = 16 - 4 = 12.$$

Therefore $$ae = \sqrt{12} = 2\sqrt{3} = 3.46^+.$$

Therefore $$e = \frac{2\sqrt{3}}{4} = \frac{\sqrt{3}}{2} = .87^-.$$

The foci are therefore at the distance $2\sqrt{3}$ from the center, and the directrices are at the distance $\frac{a}{e} = \frac{8}{\sqrt{3}} = \frac{8\sqrt{3}}{3}$ from the center. With the help of these data the graph is easily drawn. As a check the X- and Y-intercepts should be found. When $x = 0, 4y^2 + 4y - 6 = 0, 2y^2 + 2y - 3 = 0$, $y = \frac{-1 \pm \sqrt{7}}{2} = .8$ or -1.8, approximately. When $y = 0$, $x^2 - 6x - 6 = 0$, $x = 3 \pm \sqrt{15} = 6.87$ or $-.87$, approximately. These intercepts should agree with the figure.

EXERCISES

Draw the graphs of the following equations, and locate vertices, foci, and directrices. Check in each case by finding the intercepts of the curve on the X- and Y-axes.

1. $4x^2 + 9y^2 - 16x + 18y - 11 = 0.$
2. $3x^2 + 9y^2 - 6x - 27y + 2 = 0.$
3. $4x^2 + y^2 - 8x + 2y + 1 = 0.$
4. $x^2 + 15y^2 + 4x + 60y + 15 = 0.$
5. $x^2 + 2y^2 + 3x + y = 0.$
6. $2x^2 + 4y^2 + x - 8y = 0.$

PROBLEMS ON THE ELLIPSE

1. Find the equation of an ellipse, given

(a) foci at $(3, 0)$ and $(-3, 0)$, one directrix $x = 4$;
(b) foci at $(1, 1)$ and $(-1, 1)$, one directrix $x = 2$;
(c) major axis $= 8$, foci $(4, 3)$ and $(-2, 3)$;
(d) major axis $= 2$, foci $(0, \frac{1}{2})$ and $(0, -\frac{1}{2})$.

2. Find the equation of the locus of a point which moves so that the sum of its distances from the points $(3, 0)$ and $(-3, 0)$ is constantly equal to 10. What kind of curve is this locus? Draw it.

3. Find the equation of the locus of a point which moves so that the sum of its distances from the points $(c, 0)$ and $(-c, 0)$ is constantly equal to $2a$. $(a > c)$ *Ans.* $\frac{x^2}{a^2} + \frac{y^2}{a^2 - c^2} = 1.$

This result, being the equation of an ellipse, establishes the following

THEOREM. *The locus of a point which moves so that the sum of its distances from two fixed points is a constant greater than the distance between the points is an ellipse having the fixed points for foci and the constant distance for its major axis* (cf. Ex. 22, p. 96).

4. Prove the theorem of Ex. 3 by showing that the distances from any point (x_1, y_1) on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ to the foci are $a + ex_1$ and $a - ex_1$, so that their sum is the constant $2a$.

5. Use the theorem of Ex. 3 to construct an ellipse by continuous motion, with the help of a piece of string and two thumbtacks.

6. Given an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. $(a > b.)$ Let b be gradually increased until it finally equals a. How will the ellipse change? What will happen to the foci? to the eccentricity? to the directrices?

7. The lines joining a point on an ellipse with the ends of the minor axis meet the major axis in S and T. Prove that the semi-major axis is the mean proportional between OS and OT, O being the center of the ellipse.

8. A and A' are the ends of the major axis of an ellipse; P is any point on the curve, and PM and PN are perpendiculars to PA and PA' respectively, M and N being on the major axis. Prove that MN is equal to the latus rectum.

9. The same construction as in Ex. 8, except that perpendiculars are drawn to PA and PA' at A and A' respectively. Let these perpendiculars intersect in Q. Prove that the locus of Q is an ellipse whose semi-axes are a and $\frac{a^2}{b}$.

10. If a point P moves around an ellipse, starting from one end of the major axis, its distance from the center will decrease until it reaches the end of the minor axis.

11. The circle on any focal radius as diameter is tangent to the circle on the major axis as diameter.

THE HYPERBOLA

99. A third important locus is that of a point which moves so as to be always twice as far from a fixed point as from a fixed line (cf. Ex. 18, p. 96). This locus should now be constructed carefully, enough points being located so that the shape of the curve is clear. The locus is called a *hyperbola*. It would also be a hyperbola if we changed the word "twice," in the first sentence of this paragraph, to "three times," or "$1\frac{1}{2}$ times," or any other number greater than 1. The complete definition of a hyperbola is accordingly:

A hyperbola is the locus of a point whose distance from a fixed point is to its distance from a fixed line in a constant ratio greater than one. The fixed point is called the *focus*, the fixed line the *directrix*, and the constant ratio the *eccentricity*.

The student should now construct carefully several hyperbolas, using different values of the eccentricity,—for instance, $e=3$, $e=4$, $e=1\frac{1}{2}$, $e=1\frac{1}{4}$, $e=5$. Notice how the shape of the curve is affected by increasing or diminishing the eccentricity. (In drawing these curves be sure to take account of the *whole* of the locus, and not only of part of it.)

100. Equation of the hyperbola. Since the definition of the hyperbola differs from that of the ellipse only in the fact that the eccentricity is *greater* than one instead of being *less* than one, the equation of the hyperbola can be obtained in exactly the same way. The details are left to the student to carry out, and care should be taken that the figure drawn corresponds to the facts. For instance, the points A and A' will now be on opposite sides of the directrix; but we obtain, exactly as for the ellipse, the facts that

$$OB=\frac{a}{e}, \qquad OF=ae$$

(using the same letterings as for the case of the ellipse, Fig. 83). That is, the coördinates of the focus are $(ae, 0)$, and the equation of the directrix is $x=\frac{a}{e}$ (do not overlook the bearing of the fact that $e>1$).

The equation of the hyperbola takes accordingly the same form as that of the ellipse, in (9), p. 124, namely,

$$x^2(1-e^2)+y^2=a^2(1-e^2).$$

If we wish to abbreviate this, however, as we did in the case of the ellipse, we cannot do as we did before and put $b^2=a^2(1-e^2)$, because $1-e^2$ is *negative* in the case of the hyperbola. We may, however, let $b^2=a^2(e^2-1)$, in which case our equation takes the form

$$\frac{-b^2}{a^2}x^2+y^2=-b^2;$$

that is,

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1. \tag{1}$$

This is the simplest form of the equation of a hyperbola, and it may be considered as a *standard form* for the equation of the curve. By analyzing the equation we can obtain several important properties of the hyperbola.

101. (1) Solving (1) for y as an explicit function of x, we obtain $y=\pm\frac{b}{a}\sqrt{x^2-a^2}$. This shows that x^2 cannot be $< a^2$; for if it were, y would be complex, thus giving no point on the curve. This means that no point on the curve lies between the points where $x=a$ and where $x=-a$ (that is, between the points A and A'). The points A and A' are called the *vertices* of the hyperbola. For any value of $x>a$ or $<-a$ there are two values of y, numerically equal but of opposite sign, so that the curve is symmetrical with respect to the X-axis. Since this axis *crosses* the curve (at A and A'), it is called the *transverse axis*. As x increases indefinitely, the positive value of y also increases without limit; hence the curve extends indefinitely far from both the X- and the Y-axis.

(2) Solving equation (1) (§ 100) for x as an explicit function of y, we get

$$x=\pm\frac{a}{b}\sqrt{y^2+b^2}.$$

This shows that for *every* value of y there are two values of x, equal numerically but of opposite sign; hence the curve is symmetrical with respect to the Y-axis. This axis of symmetry is called the *conjugate* axis of the hyperbola, and it does not meet

the curve at all, since x cannot equal 0. The *length* $B'B = 2b$ is called the length of the conjugate axis, while $A'A = 2a$ is called the length of the transverse axis. The point of intersection of the transverse and conjugate axes is called the *center* and is the midpoint of every chord drawn through it (cf. footnote, p. 125).

(3) The relation among the three quantities a, b, and e, namely, $b^2 = a^2(e^2 - 1) = a^2e^2 - a^2$, may be easily remembered by noting that in a right triangle with a and b as the two legs, ae will be the hypotenuse. Thus, in Fig. 89 the triangle OAC has $OA = a$, $AC = b$, and hence $OC = ae$. This fact enables us to find the focus of a hyperbola when the transverse and conjugate axes are known.

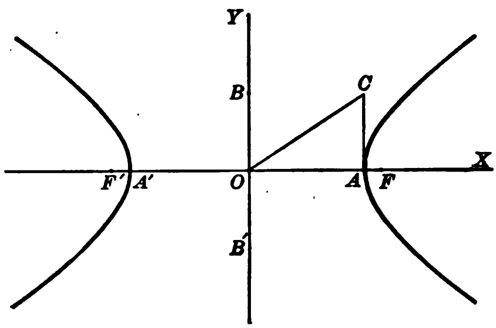


FIG. 89

(4) The chord passing through the focus, perpendicular to the transverse axis, is called the latus rectum. The student may show that its length is $\frac{2b^2}{a}$.

(5) As we have already noticed (pp. 87, 90) in the preliminary study of functions whose graphs were hyperbolas, we may expect to find that a hyperbola has two asymptotes intersecting at the center of the curve. We have not yet seen how to locate them exactly, however. This is accomplished by the following

THEOREM. *The equations of the two asymptotes of the hyperbola* $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ *are obtained by replacing the 1 by 0 in the equation of the curve, thus:*

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0.$$

Proof. The graph of the equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = \left(\frac{x}{a} + \frac{y}{b}\right)\left(\frac{x}{a} - \frac{y}{b}\right) = 0$ consists of the two straight lines

$$\frac{x}{a} + \frac{y}{b} = 0 \quad \text{and} \quad \frac{x}{a} - \frac{y}{b} = 0. \qquad \text{(Ex. 14, p. 110)}$$

These are the lines through the origin with slopes $-\frac{b}{a}$ and $\frac{b}{a}$ respectively. (Note that the latter line is in fact the line OC of Fig. 89, of course produced indefinitely.) To prove that the line $\frac{x}{a} + \frac{y}{b} = 0$ (that is, $bx + ay = 0$) is an asymptote to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, it is sufficient to show that as the point (x_1, y_1) recedes indefinitely along the curve, the distance from the line $bx + ay = 0$ to the point (x_1, y_1) approaches 0 more and more closely. Now the distance from the line $bx + ay = 0$ to the point (x_1, y_1) is

$$d = \frac{bx_1 + ay_1}{\sqrt{a^2 + b^2}}. \qquad \text{(Formula (4), p. 106)}$$

By the equation of the hyperbola,

$$b^2x_1^2 - a^2y_1^2 = a^2b^2;$$

that is,
$$bx_1 + ay_1 = \frac{a^2b^2}{bx_1 - ay_1}.$$

Replacing $bx_1 + ay_1$ in d by $\frac{a^2b^2}{bx_1 - ay_1}$,

$$d = \frac{a^2b^2}{(bx_1 - ay_1)\sqrt{a^2 + b^2}}.$$

Since the line $bx + ay = 0$ passes through the second and fourth quadrants, we are concerned with points (x_1, y_1) for which *either* $x_1 < 0$ and $y_1 > 0$, *or else* $x_1 > 0$ and $y_1 < 0$. In either case, as x_1 and y_1 increase indefinitely in numerical value, $bx_1 - ay_1$ increases indefinitely also, and hence d approaches 0. This proves that the line $bx + ay = 0$ is an asymptote to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. The proof for the line $bx - ay = 0$ is similar and should be carried through by the student.

102. Summarizing the facts obtained from the standard form of the equation of the hyperbola, $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, the locus of this equation is the hyperbola whose center is the origin, with transverse axis $2a$, conjugate axis $2b$, vertices $(\pm a, 0)$, focus $(ae, 0)$, directrix $x = \frac{a}{e}$ (where $a^2e^2 = a^2 + b^2$), and asymptotes $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0$.

Thus, the equation $\frac{x^2}{4} - \frac{y^2}{5} = 1$ has as its locus the hyperbola whose center is the origin, transverse axis 4, conjugate axis $2\sqrt{5}$, and vertices $(\pm 2, 0)$. $a^2e^2 = a^2 + b^2 = 4 + 5$. Therefore $ae = 3$,

$e=\frac{3}{2}$, $\frac{a}{e}=\frac{4}{3}$. Hence the focus is the point (3, 0), and the directrix is the line $x=\frac{4}{3}$. The asymptotes are given by the equation $\frac{x^2}{4}-\frac{y^2}{5}=0$, and are the lines through the origin with slopes $\pm\frac{\sqrt{5}}{2}$.

If in deriving the equation of the hyperbola we had chosen the line through the focus, perpendicular to the directrix, as Y-axis instead of as X-axis, the effect would have been merely to interchange x and y in the equation (1), giving

$$\frac{y^2}{a^2}-\frac{x^2}{b^2}=1. \tag{2}$$

Thus, the equation $\frac{y^2}{4}-\frac{x^2}{5}=1$ has for its locus the hyperbola with vertices $(0, \pm 2)$, focus (0, 3), directrix $y=\frac{4}{3}$, and asymptotes $\frac{y^2}{4}-\frac{x^2}{5}=0$. It is in fact the same hyperbola as that of the example above, merely rotated about the origin through an angle of 90°. Draw the figure.

EXERCISES

Draw the graphs of the following equations, and locate focus, directrix, and asymptotes:

1. $\frac{x^2}{4}-\frac{y^2}{9}=1.$
2. $\frac{x^2}{16}-\frac{y^2}{9}=1.$
3. $\frac{x^2}{4}-y^2=1.$
4. $x^2-9y^2=4.$
5. $3x^2-y^2=5.$
6. $x^2-y^2=2.$
7. $\frac{x^2}{144}-\frac{y^2}{25}=1.$
8. $\frac{y^2}{4}-\frac{x^2}{9}=1.$
9. $\frac{y^2}{4}-\frac{x^2}{12}=1.$
10. $y^2-x^2=2.$
11. $3y^2-4x^2=1.$
12. $\frac{y^2}{9}-\frac{x^2}{4}=1.$

Compare the graph of Ex. 12 carefully with that of Ex. 1, and note that they have the same asymptotes, the *transverse* axis of the one coinciding with the *conjugate* axis of the other. Two such hyperbolas are called *conjugate hyperbolas*. Draw them both in the same figure.

13. $\frac{y^2}{9}-\frac{x^2}{7}=1$. What is the equation of the conjugate hyperbola? Draw them both in the same figure.

14. $4x^2-3y^2=12$. What is the equation of the conjugate hyperbola? Draw both curves.

15. If e and e' are the eccentricities of two conjugate hyperbolas, prove that $\frac{1}{e^2} + \frac{1}{e'^2} = 1$.

16. Prove that the point $(-ae, 0)$ is also a focus, and the line $x = -\frac{a}{e}$ is a directrix, of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

***103. Generalized standard equation of the hyperbola.** As in the case of the parabola and the ellipse, so the simple standard form of the equation of a hyperbola can be generalized to apply to any hyperbola with axes parallel to the X- and Y-axes. The student should work through the details and obtain the result that the equation of the hyperbola whose center is the point (α, β), whose semi-transverse axis is a, and whose semi-conjugate axis is b is

$$\frac{(x-\alpha)^2}{a^2} - \frac{(y-\beta)^2}{b^2} = 1 \tag{3}$$

if the transverse axis is parallel to the X-axis, and

$$\frac{(y-\beta)^2}{a^2} - \frac{(x-\alpha)^2}{b^2} = 1 \tag{4}$$

if the transverse axis is parallel to the Y-axis.

The graphs of equations (3) and (4) can be thought of as the *same curves* as (1) and (2) respectively, merely translated from the position in which the origin is the center to the position in which the point (α, β) is the center.

By using (3) and (4) as standard forms the graphs of many equations can be recognized and drawn very easily.

Example. $9x^2 - 4y^2 - 18x + 24y - 63 = 0$.

This equation can be reduced to one of the forms (3) or (4) by completing the square of the x-terms and of the y-terms. Hence we write the equation

$$9(x^2 - 2x) - 4(y^2 - 6y) = 63$$

and note that to complete the square of the x-terms we must add 1 inside the first parenthesis, and that to complete the square of the y-terms we must add 9 inside the second parenthesis. This gives

$$9(x^2 - 2x + 1) - 4(y^2 - 6y + 9) = 63 + 9 - 36 = 36;$$

that is,

$$9(x-1)^2 - 4(y-3)^2 = 36,$$

or

$$\frac{(x-1)^2}{4} - \frac{(y-3)^2}{9} = 1,$$

which is in the form (3), with $\alpha = 1$, $\beta = 3$, $a = 2$, $b = 3$. Its graph is accordingly the hyperbola whose center is the point (1, 3), whose transverse axis is parallel to the X-axis and of length 4, and whose conjugate axis is of length 6. The vertices are accordingly the points (3, 3) and (−1, 3). Moreover, $a^2e^2 = a^2 + b^2 = 13$. Therefore $ae = \sqrt{13}$. Hence the distance from the center to either focus is $\sqrt{13}$. Further,

$$e = \frac{\sqrt{13}}{2};$$

hence
$$\frac{a}{e} = \frac{4}{\sqrt{13}} = \frac{4}{13}\sqrt{13} = 1.11^-,$$

the distance from the center to either directrix. The asymptotes are the lines through the center, that is, through the point (1, 3), with slopes $\frac{3}{2}$ and $-\frac{3}{2}$. What are their equations? With these data, the graph is readily drawn. As a check the intercepts should be found. When $x = 0$, $4y^2 - 24y + 63 = 0$, an equation which has complex roots ($b^2 - 4ac$ being negative); hence the graph does not meet the Y-axis. When $y = 0$, $9x^2 - 18x - 63 = 0$; that is, $x^2 - 2x - 7 = 0$, $x = 1 \pm \sqrt{8} = 3.83,\ -1.83$, approximately. These intercepts agree with the figure if it is correctly drawn.

EXERCISES

Draw the graphs of the following equations, and locate vertices, foci, directrices, and asymptotes.

1. $x^2 - y^2 - 2x + 8y - 3 = 0.$
2. $x^2 - 2y^2 + 10y = 0.$
3. $4x^2 - y^2 + 8x - 2y - 1 = 0.$
4. $x^2 - 5y^2 + 6x - 10y = 0.$
5. $3x^2 - y^2 + 12x + 2y + 14 = 0.$
6. $2x^2 - 3y^2 + x + y + 10 = 0.$

PROBLEMS ON THE HYPERBOLA

1. Find the equation of a hyperbola, given
   (a) foci at (3, 0) and (− 3, 0) and directrix $x = 1$.
   (b) foci at (0, 2) and (0, − 2) and directrix $y = \frac{1}{2}$.
   (c) transverse axis = 3, foci (2, 3), and (− 2, 3).
   (d) vertex (4, 0), asymptotes $y = 3x$, and $y = -3x$.

2. Find the equation of the locus of a point which moves so that the difference of its distances from the points (5, 0) and (− 5, 0) is equal to 6. What kind of curve is this locus?

3. Find the equation of the locus of a point which moves so that the difference of its distances from the points $(c, 0)$ and $(-c, 0)$ is constantly equal to $2a$. $(a < c)$ *Ans.* $\dfrac{x^2}{a^2} - \dfrac{y^2}{c^2 - a^2} = 1.$

This result, being the equation of a hyperbola, establishes the following

THEOREM. *The locus of a point which moves so that the difference of its distances from two fixed points is a positive constant less than the distance between the points is a hyperbola having the fixed points for foci and the constant distance for its transverse axis* (cf. Ex. 23, p. 96).

4. Prove the theorem of Ex. 3 by showing that the distances from any point (x_1, y_1) on the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ to the foci are $ex_1 + a$ and $ex_1 - a$, so that their difference is the constant $2a$.

This theorem suggests a mechanical construction of a hyperbola by continuous motion, thus: fasten pegs or thumb tacks at the foci, then pass around both a string whose ends are held together. If now a pencil point be fastened at P and both ends of the string be pulled down together, the point P will move along an arc of a hyperbola, because $PF' - PF$ will remain constant.

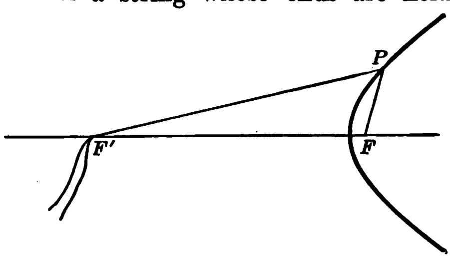


FIG. 90

5. What is the eccentricity of a hyperbola in which $a = b$? Such a hyperbola is called *equilateral.*

6. What is the angle between the asymptotes of an equilateral hyperbola?

7. Show that the foci of the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ and those of its conjugate all lie on the circle whose equation is $x^2 + y^2 = a^2 + b^2$.

8. Show that the circle of Ex. 7 meets either of the two hyperbolas on the directrix of the other.

9. The latus rectum of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is extended by the amount k so that it just reaches the asymptote. Prove that k is equal to the radius of the circle inscribed in the triangle formed by the asymptotes and the line $x = a$.

10. Prove that the foot of the perpendicular from a focus of a hyperbola upon an asymptote lies on the directrix corresponding to that focus and also upon the circle described upon the transverse axis as diameter, and that its length is equal to the semi-conjugate axis.

11. The distance of any point of an equilateral hyperbola from the center is the mean proportional to its distances from the foci.

12. Through any two points P and Q of a hyperbola, lines are drawn parallel to the asymptotes, forming the parallelogram $PRQS$. Prove that the diagonal RS passes through the center.

13. If a straight line cuts a hyperbola at the points P and P', and its asymptotes at R and R', prove that the mid-point of PP' will also be the mid-point of RR'.

14. If a point moves along a hyperbola, the product of its distances from the two asymptotes remains constant.

104. The curves that have been studied in this chapter and the preceding — namely, parabola, ellipse, hyperbola, and circle — are called *conic sections*, or simply *conics*, because they can all be obtained as plane sections of a circular cone. They were originally studied from that point of view, and nearly all their elementary properties that are known to-day were proved by the Greek geometers more than two thousand years ago.[1] The conic sections are of especial interest because of the fact that the paths of all the heavenly bodies are curves of this kind. This fact was first established by the great German astronomer and mathematician Johannes Kepler, in 1609. He showed that the planet Mars moves in an ellipse; the other planets, including of course the earth, do the same, while many comets move in parabolas or hyperbolas.

[1] The most complete study of the conic sections among the Greeks was made by Apollonius of Perga, about 200 B.C.

105. From the point of view of their equations it will be noted that all these curves have one thing in common: the equation of every conic section is *of the second degree* in x and y; that is, when cleared of fractions and radicals and reduced to its simplest form, each of our standard equations has been of the second degree. We cannot as yet prove that *every* conic section must have as its equation one of the second degree, but this is a fact, and the proof of it will be possible at a later stage of the mathematical course. The converse statement, that every equation of the second degree in x and y has for its locus a conic section, is not true, because many equations of the second degree have no locus at all. If there is a locus, however, it must be either a conic section or something simpler — a pair of straight lines (as $x^2 - y^2 = 0$), a single straight line (as $x^2 - 2xy + y^2 = 0$), or a point (as $x^2 + y^2 = 0$). This assertion also will not be proved here.

The most general form of equation of the second degree in x and y is

$$ax^2 + bxy + cy^2 + dx + ey + f = 0,$$

where at least one of the coefficients a, b, c is different from 0. The student should show how to get each one of the standard forms of equation of the conics by giving special values to a, b, c, d, e, and f. For example, the circle $x^2 + y^2 = r^2$ is obtained if we put $a = 1$, $b = 0$, $c = 1$, $d = e = 0$, $f = -r^2$.

CHAPTER VIII

SIMULTANEOUS EQUATIONS

106. Review questions. Give the standard forms of the equations of straight line, circle, parabola, ellipse, and hyperbola. How are the asymptotes of a hyperbola determined? How can the coördinates of the point of intersection of two straight lines be found? of a straight line and a parabola?

107. This last question brings us to the problem of this chapter, which is, to find the points of intersection of a straight line and any conic section, or the points of intersection of two conics. Some important geometric results can then be obtained, based on the solutions of these problems. As a preliminary step the problems on page 45 should be reviewed, that there may be no difficulty in the algebraic solution of a quadratic equation in one unknown quantity.

108. Intersection of straight line and conic. The same method which was used in Chapter III, to find the intersections of a straight line and a parabola, can be used in the case of a straight line and any conic section. An example will make this clear.

Example. Find the points of intersection of the straight line

$$x + y = 2, \tag{1}$$

and the conic

$$4x^2 + y^2 = 4. \tag{2}$$

Making the graphs of (1) and (2), we see that their intersections are the points A and C (Fig. 91). A is evidently $(0, 2)$, and C is not far from $(\frac{3}{4}, 1\frac{1}{3})$.

To determine algebraically the exact values of the coördinates of A and C (A we have found exactly, because $(0, 2)$ satisfies both equations (1) and (2), but $(\frac{3}{4}, \frac{4}{3})$ does not satisfy the equations) we obtain from (1) the value of y as a linear function of x, thus:

$$y = 2 - x.$$

Substituting this value of y in (2),

$$4x^2 + (2 - x)^2 = 4. \tag{3}$$

Since (3) has been obtained by using *both* (1) and (2), the values of x which it determines will be the abscissas of points lying both on (1) and on (2); in other words, the roots of equation (3) are the abscissas of A and C, the points of intersection of (1) and (2).

Simplifying (3),

$$5x^2 - 4x = 0. \qquad (4)$$

$$x(5x - 4) = 0.$$

Therefore $\qquad x = 0$ or $\frac{4}{5}$.

Hence the abscissas of A and C, the points of intersection of the straight line and the ellipse, are 0 and $\frac{4}{5}$ respectively. Since $x + y = 2$, $x = 0$ corresponds to $y = 2$, and $x = \frac{4}{5}$ to $y = \frac{6}{5}$. Therefore (0, 2) and $(\frac{4}{5}, \frac{6}{5})$ are the points of intersection.

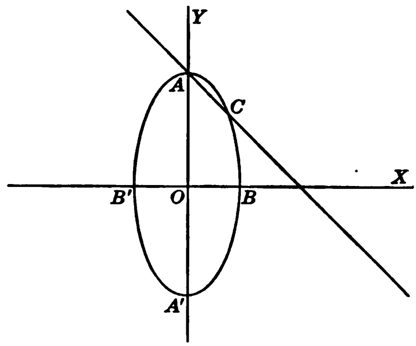


FIG. 91

Evidently this method can be used in any case where it is desired to find the points of intersection of a straight line and a conic. We may solve the linear equation for y as a function of x (or for x as a function of y) and substitute in the equation of the conic, getting a quadratic equation in the one variable x (or y), whose roots will be the abscissas (or ordinates) of the required points of intersection. The other coördinate is then found by substituting in the linear equation the value of the coördinate that has been found. (The equation of the conic should not be used for this last substitution, because it would usually give two values where only one is correct. Thus, in the example above, if we had substituted $x = 0$ in the equation of the *ellipse*, we should have found $y = +2$ or -2, whereas $y = +2$ is the correct value.)

EXERCISES

Solve both graphically and algebraically the following pairs of simultaneous equations:

1. $\begin{cases} x^2 + y^2 = 25, \\ x - y = 1. \end{cases}$

2. $\begin{cases} y = 2x^2 - 3x - 4, \\ y - x = 2. \end{cases}$

3. $\begin{cases} y = 3x - 3x^2 + 7, \\ 3x - 2y + 5 = 0. \end{cases}$

4. $\begin{cases} x^2 + 4y^2 = 25, \\ 2x - y = 4. \end{cases}$

5. $\begin{cases} 3x^2 + 2y^2 = 11, \\ x - 3y = 7. \end{cases}$

6. $\begin{cases} \dfrac{x^2}{25} + \dfrac{y^2}{9} = 1, \\ 2x - y = 14. \end{cases}$

7. $\begin{cases} 3x^2 + 16y^2 = 192, \\ x + 2y = 10. \end{cases}$

8. $\begin{cases} x^2 - y^2 = 1, \\ x - 2y + 1 = 0. \end{cases}$

9. $\begin{cases} x^2 - 2y^2 = 4, \\ 3x - 2y = 10. \end{cases}$

10. $\begin{cases} \dfrac{x^2}{36} - \dfrac{y^2}{20} = 1, \\ x - y = 4. \end{cases}$

11. $\begin{cases} x - 3y = 1, \\ xy + y^2 = 5. \end{cases}$

12. $\begin{cases} 4y = 5x + 1, \\ 2xy = 33 - x^2. \end{cases}$

13. $\begin{cases} 7x^2 - 8xy = 159, \\ 5x + 2y = 7. \end{cases}$

14. $\begin{cases} xy + 3y^2 = 42, \\ 2x + y = 13. \end{cases}$

109. Tangent to a Conic. Just as in the case of the parabola (p. 49), so here for any conic, the sign of the discriminant of the quadratic equation corresponding to (3), p. 141, enables us to tell whether the straight line (1) and the conic (2) have two points in common, or only one, or none. In case they have only one common point (which happens when the discriminant equals zero), the line will, in general, be *tangent* to the conic.

Example. For what values of c will the line

$$3x + 4y = c \tag{1}$$

be tangent to the curve

$$x^2 + y^2 = 25\,? \tag{2}$$

Solving (1) for y, $\qquad y = \dfrac{c - 3x}{4}.$

Substituting this value of y in (2),

$$x^2 + \left(\frac{c - 3x}{4}\right)^2 = 25;$$

that is,

$$25x^2 - 6cx + c^2 - 400 = 0. \tag{3}$$

The roots of (3) are the abscissas of the points where (1) meets (2). If the line is *tangent*, these abscissas must be *equal*, and this requires that the discriminant of (3) shall equal zero. The discriminant of (3) is

$$\begin{aligned} D &= (-6c)^2 - 4 \cdot 25(c^2 - 400) \\ &= 36c^2 - 100c^2 + 40{,}000 = 40{,}000 - 64c^2. \end{aligned}$$

If $D = 0$, $\qquad 64c^2 = 40{,}000,$

$$c^2 = 625.$$

Therefore $\qquad c = \pm 25.$

Hence the line $3x + 4y = 25$ or $3x + 4y = -25$ will be tangent to the circle (2). (Verify this by a figure.) If $c^2 > 625$, that is, if $c > 25$ or < -25, the value of the discriminant will be *negative*, and the straight line will not meet the circle at all. If $c^2 < 625$, that is, if $-25 < c < 25$, the value of the discriminant will be *positive*, and the straight line will meet the circle in two distinct points. Each of these possibilities should be illustrated by a figure.

EXERCISES

Determine for what values of k the following lines and conics will be tangent, and discuss the values of k that will give intersection or nonintersection of the two graphs:

1. $\begin{cases} x^2 + y^2 = 9, \\ 3y = 4x + k. \end{cases}$

2. $\begin{cases} x^2 - y^2 = 9, \\ 5x - 4y = k. \end{cases}$

3. $\begin{cases} \dfrac{x^2}{16} + \dfrac{y^2}{25} = 1, \\ kx + 4y = 20. \end{cases}$

4. $\begin{cases} y = x^2 - 3x - 4, \\ x + 2y = k. \end{cases}$

5. $\begin{cases} \dfrac{x^2}{36} + \dfrac{y^2}{25} = 1, \\ 5x + ky = 60. \end{cases}$

6. $\begin{cases} y^2 = 4x, \\ y = 2x + k. \end{cases}$

7. $\begin{cases} y^2 = 4x, \\ kx + y = 3. \end{cases}$

8. $\begin{cases} 2x^2 - 3y^2 = 5, \\ 4x + ky = 5. \end{cases}$

9. $\begin{cases} \dfrac{x^2}{9} - \dfrac{y^2}{4} = 1, \\ 6x - ky = 9. \end{cases}$

10. $\begin{cases} y^2 = 4x + 8, \\ x + ky = 2. \end{cases}$

11. $\begin{cases} 4x^2 + 9y^2 = 36, \\ y = \frac{1}{3}x + k. \end{cases}$

12. $\begin{cases} y^2 = 4x, \\ y = mx + k \text{ (m a fixed number).} \end{cases}$

13. $\begin{cases} y^2 = 4ax \text{ (a a fixed number),} \\ y = mx + k \text{ (m a fixed number).} \end{cases}$ *Ans.* $k = \dfrac{a}{m}$.

This result means that the line $y = mx + \dfrac{a}{m}$ is tangent to the parabola $y^2 = 4ax$ for *any* given value of m (not 0).

14. Find the coördinates of the point of contact of the tangent line of Ex. 13. *Ans.* $\left(\dfrac{a}{m^2}, \dfrac{2a}{m}\right)$.

15. Find the Y-intercept of the tangent line of Ex. 13. Draw a conclusion from the results of the preceding exercise and this one (cf. Exs. 5, 6, p. 52).

16. Find the X-intercept of the tangent line of Ex. 13. Compare this with the abscissa of the point of contact.

17. Use the results of the preceding problems to find the equation of the tangent to the parabola $y^2 = 6x$ at the point $(\frac{8}{3}, 4)$; at the point $(6, 6)$; at the point (x_1, y_1).

18. Prove that the tangents to the parabola $y^2 = 4\,ax$ at the ends of the latus rectum are perpendicular to each other and intersect on the directrix.

19. Give a geometric construction for the tangent to a parabola at any given point.

20. Discover a construction for the two tangents to a parabola from an external point.

21. The *normal* to a curve means the perpendicular to the tangent, drawn through the point of contact. Find the equation of the normal to the parabola $y^2 = 4\,ax$, at the point (x_1, y_1).

22. Prove that the *subnormal* is constant for all points on the parabola $y^2 = 4\,ax$. (The subnormal is the distance from M, the foot

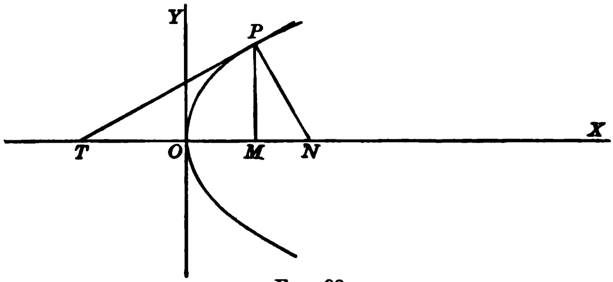


FIG. 92

of the perpendicular from the point of contact to the X-axis, to N, the intersection of the normal with the X-axis. See Fig. 92.)

23. Prove that the tangent to a parabola bisects the angle between the focal radius at the point of contact and the line parallel to the axis through the same point.

24. For what values of k will the line $y = mx + k$ (m a fixed number) be tangent to the ellipse $4\,x^2 + 9\,y^2 = 36$?

25. Answer the same question for the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and the line $y = mx + k$ (m a fixed number). (*Ans.* $k = \pm\sqrt{b^2 + a^2m^2}$.) This result shows that the line $y = mx \pm \sqrt{b^2 + a^2m^2}$ is tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ for *any* given value of m. The double sign shows that there are *two* such tangents, that is, that there are two tangents with any given slope m, — a fact which is geometrically self-evident.

26. Find the coördinates of the point of contact of the tangent line of Ex. 25.

110. Construction of a tangent to an ellipse. The results of Exs. 25 and 26 enable us to construct the tangent to an ellipse, thus:

The tangent line PT (Fig. 93) has the equation

$$y = mx + \sqrt{b^2 + a^2m^2}. \quad (1)$$

This meets the X-axis at the point T, whose abscissa is the X-intercept of (1). This is found by setting $y = 0$ in (1), giving

$$0 = mx + \sqrt{b^2 + a^2m^2},$$

or $x = -\dfrac{\sqrt{b^2 + a^2m^2}}{m}$;

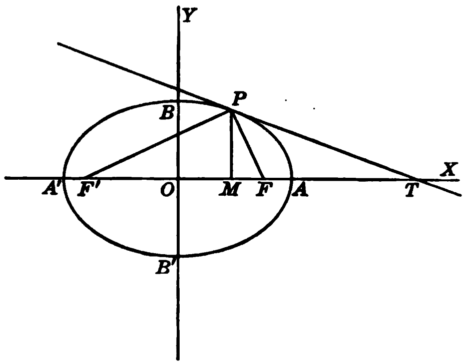


FIG. 93

that is, $$OT = -\frac{\sqrt{b^2 + a^2m^2}}{m}.$$

But, by Ex. 26, $$OM = \frac{-a^2m}{\sqrt{b^2 + a^2m^2}}.$$

Therefore $$OT \cdot OM = a^2, \quad \text{or} \quad OT = \frac{a^2}{x_1},$$

where $x_1 = OM =$ the abscissa of the point of contact.

This is a very important and remarkable result, because it shows that the distance OT *is independent of* b, that is, *does not depend upon the minor axis of the ellipse.* In other words, if we have a set of ellipses, all having the same major axis $A'A$ but different minor axes, the tangent to *any one* of the ellipses, at the point whose abscissa is x_1, will pass through T. If $b = a$, the ellipse becomes the circle with

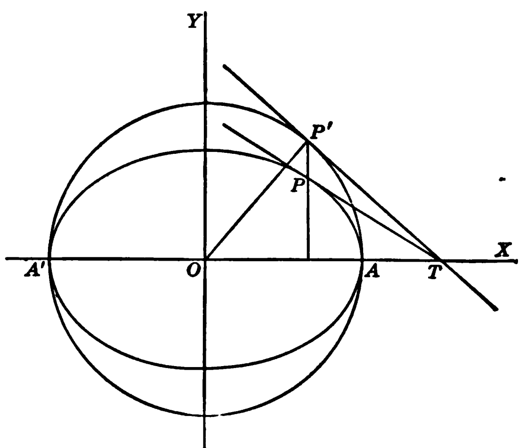


FIG. 94

$A'A$ as diameter, and hence its tangent (at P', the point whose abscissa is x_1) is easily constructed. It is the line $P'T$, perpendicular to OP' (Fig. 94). The intersection of this tangent with the axis $A'A$ produced is the common point T where the tangents to *all* the ellipses meet the axis. Hence we only need to join this point T with the point P on the ellipse to get the required tangent TP.

PROBLEMS

1. In Fig. 93, prove that $F'T : FT = F'P : PF$, and hence that PT bisects the exterior angle of the triangle FPF'. This theorem gives another construction for the tangent to an ellipse at any point.

2. Obtain the equation of the tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at the point (x_1, y_1) in the form $\frac{x_1 x}{a^2} + \frac{y_1 y}{b^2} = 1$.

HINT. It passes through (x_1, y_1) and $\left(\frac{a^2}{x_1}, 0\right)$.

3. Find the equation of the normal to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, at the point (x_1, y_1).

4. Find the X-intercept of the normal of the preceding problem.

5. Prove that OF, the distance from the center to the focus of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, is the mean proportional between the X-intercepts of the tangent and the normal.

6. State and prove the corresponding theorem for the Y-intercepts of the tangent and the normal.

7. Construct the two tangents from an external point to an ellipse.

A similar set of problems can be worked out for the hyperbola, but the results do not show enough difference from those for the ellipse to justify stating them explicitly here. This whole subject is considered from a higher point of view in the chapter, Introduction to the Differential Calculus.

111. Intersection of two conics. To obtain graphically the coördinates of the points of intersection of two conics is a simple matter; unfortunately the algebraic method, by which alone

we could be certain of obtaining *exact* results, is not practicable[1] except in a comparatively few special types of problem. In each of the following exercises the graphical solution will be found simple, in some cases even leading to exact results.

EXERCISES

Find (graphically) the coördinates of the points of intersection of each of the following pairs of conics:

1. $\begin{cases} x^2 + y = 7, \\ x + y^2 = 11. \end{cases}$

2. $\begin{cases} x^2 - y = 17, \\ x - y^2 = 3. \end{cases}$

3. $\begin{cases} xy = 10, \\ x^2 - y^2 - 4x = 1. \end{cases}$

4. $\begin{cases} x^2 + 3y^2 = 7, \\ y^2 + 2y = \dfrac{x}{2}. \end{cases}$

112. Solvable type; both equations homogeneous. If we have two equations of conics in which *every term containing the variables is of the second degree*, then algebraic solution is always possible. The most general form of such an equation is

$$ax^2 + bxy + cy^2 = d,$$

which is called a *homogeneous* equation of the second degree in x and y. When each of the given equations is of this type, we can eliminate the constant term and get an equation of the form

$$a'x^2 + b'xy + c'y^2 = 0,$$

which can be factored if the points of intersection have rational coördinates, and often even when they do not.

Example.
$$\begin{cases} 2x^2 - 3xy + 5y^2 = 14, & (1) \\ x^2 + 4xy - 2y^2 = 19. & (2) \end{cases}$$

To eliminate the constant term, multiply (1) by 19, and (2) by 14, and subtract. The result is

$$24x^2 - 113xy + 123y^2 = 0. \qquad (3)$$

[1] That is, it would involve more advanced algebraic work than we are yet prepared for.

Solving (3) for x as a function of y, we get

$$x = \frac{113\,y \pm \sqrt{12769\,y^2 - 11808\,y^2}}{48}$$
$$= \frac{113\,y \pm \sqrt{961\,y^2}}{48}$$
$$= \frac{113\,y \pm 31\,y}{48}$$
$$= 3\,y \text{ or } \tfrac{41}{24}\,y.$$

(Equation (3) could of course have been factored, thus:

$$24\,x^2 - 113\,xy + 123\,y^2 = (x - 3\,y)(24\,x - 41\,y) = 0.$$

Therefore $\qquad x = 3\,y$ or $\tfrac{41}{24}\,y$.)

(a) Using $x = 3\,y$, we get, from (1),

$$2\,(3\,y)^2 - 3\,y\,(3\,y) + 5\,y^2 = 14,$$
$$14\,y^2 = 14,$$
$$y^2 = 1,$$
$$y = \pm 1.$$

Since $x = 3\,y$, when $y = 1$, $x = 3$, and when $y = -1$, $x = -3$. Check these pairs of values by substituting them in both (1) and (2).

(b) Using $x = \tfrac{41}{24}\,y$, we get, from (1),

$$2\,(\tfrac{41}{24}\,y)^2 - 3\,y\,(\tfrac{41}{24}\,y) + 5\,y^2 = 14,$$
$$\tfrac{1645}{288}\,y^2 = 14,$$
$$y^2 = \tfrac{576}{235},$$
$$y = \pm\frac{24}{\sqrt{235}} = \pm\frac{24}{235}\sqrt{235}.$$

Since $\qquad x = \tfrac{41}{24}\,y,$

$$x = \pm\tfrac{41}{235}\sqrt{235}.$$

The graphical solution is not so simple as the algebraic one in this case, because (on account of the xy term) the equations cannot be reduced to any of the standard forms of the equations of conics. To draw the graphs we shall therefore have to compute the coördinates of enough points so that the form of each curve becomes clear. Solving (1) for y as a function of x,

$$y = \frac{3\,x \pm \sqrt{9\,x^2 - 20(2\,x^2 - 14)}}{10} = \frac{3\,x \pm \sqrt{280 - 31\,x^2}}{10}.$$

Using this, we get the table of values as follows:

x	0	1	2	3	$\pm 2.6+$
y	± 1.7	1.8 or -1.2	1.8+ or $-6+$	1 or .8	0

Negative values of x give the same results but with opposite sign. It is evident from the value of y above that x cannot be $> 3^+$. The graph is an ellipse.

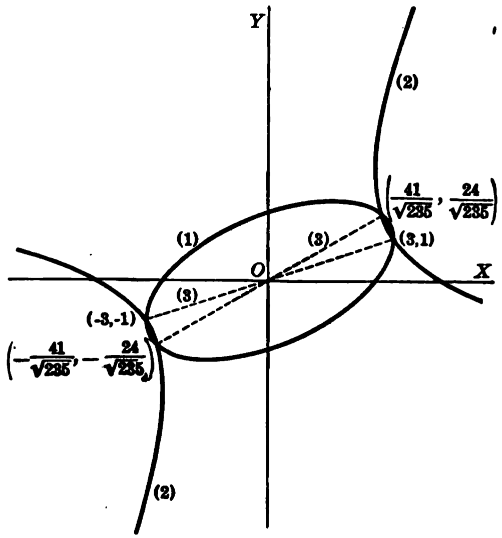


Fig. 95

Similarly, from (2),

$$y = \frac{4x \pm \sqrt{16x^2 - 8(19 - x^2)}}{4} = \frac{2x \pm \sqrt{6x^2 - 38}}{2}.$$

The table of values is as follows:

x	0	1	2	3	4	$\pm 4.3+$
y	complex			1 or 5	.2 or 7.8	0

The graph is a hyperbola. Fig. 95 shows the graphs of (1) and (2), their four intersections corresponding to the algebraic solution. The straight lines given by (3) (p. 148) are also shown.

EXERCISES

Solve for x and y, checking graphically where practicable:

1. $\begin{cases} x^2 - y^2 = 1, \\ x^2 - xy + y^2 = 1. \end{cases}$

2. $\begin{cases} x^2 - 2xy = 3, \\ xy + y^2 = 4. \end{cases}$

3. $\begin{cases} x^2 + 2xy - y^2 = 14, \\ 2x^2 + 3xy + 7y^2 = 42. \end{cases}$

4. $\begin{cases} x^2 + xy + 2y^2 = 44, \\ 2x^2 - xy + y^2 = 16. \end{cases}$

5. $\begin{cases} 2y^2 - 4xy + 3x^2 = 17, \\ y^2 - x^2 = 16. \end{cases}$

6. $\begin{cases} x^2 - xy - y^2 = 5, \\ 2x^2 + 3xy + y^2 = 28. \end{cases}$

7. $\begin{cases} x^2 - xy = 35, \\ xy + y^2 = 18. \end{cases}$

8. $\begin{cases} x^2 - xy + y^2 = 21, \\ y^2 - 2xy = -15. \end{cases}$

***113. Algebraic solution of some equations of higher degree.** In the case of simultaneous equations, one or both of which is of higher degree than the second, it is only in special cases that algebraic solution by elementary methods is possible. In some cases, however, it is so very simple that it is worthy of consideration. As for the graphical solution, we shall neglect it entirely, because it involves in nearly all cases too difficult a process of computing tables of corresponding values of x and y.

***114.** The following illustrative examples should be carefully studied, until the general methods used are well understood. It will be noticed that in each case the given equations are combined in such a way as to lead to a linear and a quadratic equation, which pair can then always be solved by the methods with which we are already familiar.

Example 1.
$$x^3 + y^3 = 133, \quad (1)$$
$$x + y = 7. \quad (2)$$

Here we notice that if (1) is divided by (2), we obtain a quadratic equation. In fact, division gives
$$x^2 - xy + y^2 = 19. \quad (3)$$

We may now solve (3) with (2) as usual, or we may proceed thus:

Squaring (2), $$x^2 + 2xy + y^2 = 49. \quad (4)$$

Subtracting (3) from (4), $$3xy = 30.$$
$$xy = 10. \quad (5)$$

Multiplying (5) by 4 and subtracting from (4),
$$x^2 - 2xy + y^2 = 9.$$
$$x - y = \pm 3. \quad (6)$$

Adding (6) and (2), $2x = 7 \pm 3 = 10$ or 4.

Therefore $x = 5$ or 2.

Therefore $y = 2$ or 5 (since $x + y = 7$).

Hence the solutions are (5, 2) and (2, 5).

Another method is to cube (2) and subtract (1) from the result. This gives $3xy(x+y) = 210$, and hence $3xy = 30$, $xy = 10$, since $x + y = 7$. From here we can proceed as usual, substituting the value of x (or y) from (2) in the equation $xy = 10$; or we can continue as from equation (5) above.

Example 2.
$$x^4 + y^4 = 706, \tag{1}$$
$$x + y = 8. \tag{2}$$

Raising (2) to the fourth power,

$$x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4 = 4096.$$

Subtracting (1), $4x^3y + 6x^2y^2 + 4xy^3 = 3390.$ (3)

Squaring (2) and multiplying by $4xy$,

$$4x^3y + 8x^2y^2 + 4xy^3 = 256xy. \tag{4}$$

Subtracting (3) from (4), $2x^2y^2 = 256xy - 3390.$

$$x^2y^2 - 128xy + 1695 = 0.$$
$$(xy - 15)(xy - 113) = 0. \tag{5}$$

We can now use (5) and (2) together, getting (5, 3) and (3, 5) as the only real solutions.

Example 3.
$$x^4 + x^2y^2 + y^4 = 481, \tag{1}$$
$$x^2 + xy + y^2 = 37. \tag{2}$$

Dividing (1) by (2), $x^2 - xy + y^2 = 13.$ (3)

Subtracting (3) from (2), $2xy = 24.$

$$xy = 12. \tag{4}$$

We can now obtain $(x+y)^2$ and $(x-y)^2$ by using (4) with (2) and (3); the remainder of the work is left to the student.

EXERCISES

Solve for x and y, and check:

1. $\begin{cases} x^3 + y^3 = 65, \\ x + y = 5. \end{cases}$

2. $\begin{cases} x^3 + y^3 = 72, \\ x + y = 6. \end{cases}$

3. $\begin{cases} x^3 + y^3 = 35, \\ xy(x+y) = 30. \end{cases}$

4. $\begin{cases} 4x^2 + xy + 4y^2 = 37, \\ 5x^2 + 5y^2 = 50. \end{cases}$

5. $\begin{cases} x^2 + y^2 + x + y = 36, \\ xy = 10. \end{cases}$

6. $\begin{cases} x^2 + y^2 - x + y = 20, \\ xy = 4. \end{cases}$

7. $\begin{cases} x^4 + y^4 = 82, \\ x + y = 4. \end{cases}$

8. $\begin{cases} x^5 - y^5 = 3093, \\ x - y = 3. \end{cases}$

9. $\begin{cases} x^2 - xy + y^2 = 7, \\ x^4 + x^2y^2 + y^4 = 133. \end{cases}$

10. $\begin{cases} x^3 - y^3 = 98, \\ xy(x - y) = 30. \end{cases}$

11. $\begin{cases} x^2 - y^2 = 14 - x, \\ y^2 = 2x + 6. \end{cases}$

12. $\begin{cases} x^2 + xy + y^2 = 19, \\ x + xy + y = 1. \end{cases}$

13. The sum of two numbers is 28, and their product is 147. Find the numbers. (Cf. also Exs. 3 and 4, p. 55.)

14. The product of two numbers is 180, and their quotient is $\frac{5}{4}$. What are they?

15. The diagonal of a field is 89 rd. long, and another field which is 3 rd. less both in length and in breadth has a diagonal 85 rd. long. What are the dimensions of each field?

16. The diagonal of a rectangle is 68 cm., and if the length were increased by 2 cm. and the breadth diminished by the same amount, the area would be diminished by 60 sq. cm. Find the dimensions.

17. A sum of money and its interest for one year amount to $13,520. If the sum is increased by $200 and the rate of interest by $\frac{1}{2}$%, the amount will be $13,794 for one year. Find principal and interest.

18. The fore wheel of a carriage turns in a mile 132 times more than the rear wheel, but if the circumferences were each increased by 2 ft., it would turn only 88 times more. Find the circumference of each.

19. A merchant buys a certain amount of wheat for $322. The market goes up 3¢, and he can then for $323 get 10 bushels less than he could before for $322. What was the price per bushel?

CHAPTER IX

FURTHER STUDY OF THE TRIGONOMETRIC FUNCTIONS. POLAR COÖRDINATES

115. Review questions. Define the trigonometric functions. Give their signs in each of the four quadrants. State four relations among the functions of an angle. What are the values of the functions of $180° - \theta$ and of $90° + \theta$ in terms of functions of θ? How can a right triangle be solved? How is the resultant of two forces found? Give the slope of the straight line joining the points (x_1, y_1) and (x_2, y_2). What is the slope of the line $lx + my + n = 0$? State the relation between the slopes of two perpendicular lines; of two parallel lines.

116. Chapter IV included a few of the simplest applications of the trigonometric functions. In this chapter we shall take up some other problems in which they are used, and also study their graphs.

117. Solution of the oblique triangle. We saw in Chapter IV how to solve any right triangle. Inasmuch as any oblique-angled triangle can be divided into two right triangles by drawing an altitude, the unknown parts can be found without the use of any new principles. A more practical method, however, is obtained if we discover the relations that exist among the sides and angles of the oblique triangle itself, thus avoiding the necessity of solving the two right triangles separately. Moreover, the study of these relations and of many others involving the trigonometric functions has very great value for its own sake and on account of its wide application in mathematical work of a more advanced character.

I. The Law of Sines

118. The following discussion applies only to Fig. 96, (*a*), which represents an acute-angled triangle.

Draw the altitude *CD*, thus forming the right triangles *BCD* and *ACD*.

In the triangle ACD, $\frac{h}{b} = \sin\alpha.$

Therefore $h = b\sin\alpha.$ (1)

In the triangle BCD, $\frac{h}{a} = \sin\beta.$

Therefore $h = a\sin\beta.$ (2)

From (1) and (2) it follows that

$$a\sin\beta = b\sin\alpha, \tag{3}$$

or, dividing by $\sin\alpha\sin\beta$,

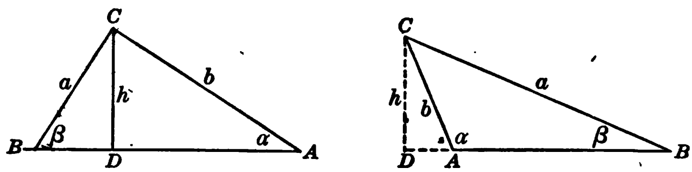
$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta}. \tag{4}$$

If we draw the altitude from B, we shall get, in the same way,

$$\frac{a}{\sin\alpha} = \frac{c}{\sin\gamma}. \tag{5}$$

Therefore $$\boldsymbol{\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma}}. \tag{6}$$

This extremely important relation is known as the *Law of*



(a) (b)

FIG. 96

Sines. It may be stated in words as follows: *In any triangle the sides are proportional to the sines of the opposite angles.*

119. We have proved only that this law is true for acute-angled triangles; in Fig. 96, (*b*), where the triangle ABC is *obtuse-angled* (α being the obtuse angle), we see that the altitude CD falls *outside* the triangle. Accordingly,

$$\frac{h}{b} = \sin\angle DAC = \sin(180° - \alpha) = \sin\alpha \text{ (p. 69).}$$

With this hint the student may complete the proof for himself, thus establishing the truth of the Law of Sines for any oblique triangle.

II. The Law of the Projections

120. Let the projections[1] of the sides a and b of the triangle ABC upon the side c be p and q respectively; then (Fig. 97, (a))

$$p = a\cos\beta \quad \text{and} \quad q = b\cos\alpha.$$

Since

$$p + q = c,$$

$$\boldsymbol{c = a\cos\beta + b\cos\alpha.} \tag{7}$$

By drawing the altitude upon the side b we get, in the same way,

$$\boldsymbol{b = a\cos\gamma + c\cos\alpha,} \tag{8}$$

and by drawing the altitude upon the side a,

$$\boldsymbol{a = b\cos\gamma + c\cos\beta.} \tag{9}$$

In the obtuse-angled triangle (Fig. 97, (b)) the side c is not the sum, but the difference, of the projections p and q; but equation (7)

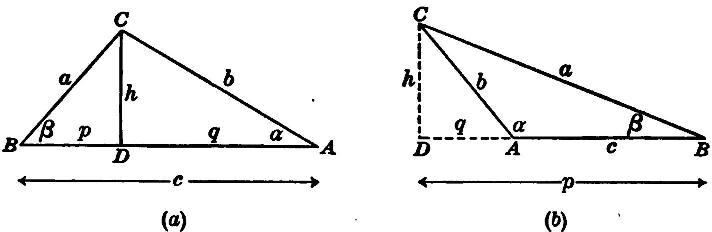


Fig. 97

is true in this case also, because $q = b\cos(180° - \alpha) = -b\cos\alpha$ (p. 69). The details should be worked out by the student.

III. The Law of Cosines

121. If we apply the Pythagorean Theorem to the right triangle BCD (Fig. 97, (a)), we have

$$a^2 = p^2 + h^2. \tag{10}$$

Replacing p by its value $c - q$,

$$\begin{aligned} a^2 &= (c-q)^2 + h^2 \\ &= c^2 - 2cq + q^2 + h^2. \end{aligned} \tag{11}$$

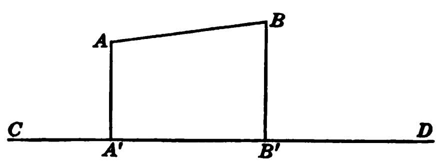


Fig. 98

[1] The projection of a line segment AB upon a line CD is the segment $A'B'$, between the feet of the perpendiculars from A and B upon CD (Fig. 98).

But $\quad q^2 + h^2 = b^2.$

Therefore $\quad a^2 = c^2 - 2\,cq + b^2. \qquad (12)$

And, finally, $\quad q = b \cos \alpha.$

Therefore $\quad \boldsymbol{a^2 = b^2 + c^2 - 2\,bc \cos \alpha}. \qquad (13)$

By starting with the triangle ACD instead of the triangle BCD we obtain, in exactly the same way,

$$\boldsymbol{b^2 = a^2 + c^2 - 2\,ac \cos \beta}. \qquad (14)$$

And by drawing the altitude from A or B we get, likewise,[1]

$$\boldsymbol{c^2 = a^2 + b^2 - 2\,ab \cos \gamma}. \qquad (15)$$

Equations (13), (14), and (15) are known as the *Law of Cosines.* This law may be stated in words as follows:

The square of any side of a triangle is equal to the sum of the squares of the other two sides, minus twice their product times the cosine of the included angle.

As in the case of the other theorems of this section, the student should carry through the proof for Fig. 97, (*b*), and show that the same relation holds true in an obtuse-angled triangle.

122. By the aid of Theorems I–III it is possible to solve any oblique triangle without first drawing an altitude and solving the auxiliary right triangles thus formed. The student should state what Laws I, II, and III give when applied to a *right* triangle.

Example 1. Given $\alpha = 36°$, $\beta = 69°$, $a = 35$ ft., to find γ, b, and c.

$$\gamma = 180° - (\alpha + \beta) = \underline{\underline{75°}}.$$

Using the Law of Sines, $\quad \dfrac{b}{\sin \beta} = \dfrac{a}{\sin \alpha};$

therefore $\quad b = \dfrac{35 \cdot \sin 69°}{\sin 36°} = \dfrac{35 \cdot 0.9336}{0.5878} = \underline{\underline{55.59}}.$

Again, $\quad \dfrac{c}{\sin \gamma} = \dfrac{a}{\sin \alpha};$

therefore $\quad c = \dfrac{a \sin \gamma}{\sin \alpha} = \dfrac{35 \cdot 0.9659}{0.5878} = \underline{\underline{57.54}}.$

Check. $\dfrac{b}{\sin \beta} = \dfrac{c}{\sin \gamma}. \quad \dfrac{55.59}{\sin 69°} = \dfrac{57.54}{\sin 75°}.$ Hence the results obtained are verified.

[1] Equations (14) and (15) are the same formula as (13), only expressed in different letters; it is logically correct to derive them from the latter by merely changing the letters suitably.

Example 2. Given $a = 15$, $b = 20$, $\gamma = 51°$, to find c, α, and β.

Using the Law of Cosines, $c^2 = a^2 + b^2 - 2\,ab\cos\gamma$

$$= 225 + 400 - 600\cos 51°$$
$$= 247.42.$$

Therefore $c = \underline{\underline{15.73}}.$

To find α, use the Law of Sines, thus:

$$\frac{\sin\alpha}{a} = \frac{\sin\gamma}{c}.$$

Therefore $\sin\alpha = \dfrac{15\sin 51°}{15.73} = .7410.$

Hence $\alpha = \underline{\underline{47°\,50'}}.$

Similarly, $\dfrac{\sin\beta}{b} = \dfrac{\sin\gamma}{c}.$

Therefore $\sin\beta = \dfrac{20\sin 51°}{15.73} = .9880.$

Hence $\beta = \underline{\underline{81°\,10'}}.$

Check. $\alpha + \beta + \gamma = 180°.$

EXERCISES

Solve the triangles in Exs. 1–16, drawing an accurate figure for each and checking both by measurement and by computation:

1. $a = 7$ in., $\beta = 68°$, $\gamma = 54°$.
2. $a = 16$ ft., $\alpha = 80°$, $\beta = 37°$.
3. $a = 62$ ft., $b = 55$ ft., $\gamma = 42°$.
4. $a = 8.6$ ft., $c = 11.3$ ft., $\beta = 70°$.
5. $b = 21.2$ ft., $\gamma = 71°$, $\beta = 50°$.
6. $a = 5$ in., $b = 6$ in., $c = 7$ in. (Use the Law of Cosines.)
7. $a = 13$, $b = 15$, $c = 16$.
8. $a = .82$, $b = .59$, $c = .71$.
9. $\gamma = 38°$, $b = 14$, $c = 20$.
10. $\beta = 19°$, $\gamma = 83°$, $b = 35.2$ ft.
11. $a = 15$, $\alpha = 150°$, $\beta = 12°$.
12. $b = 26$, $\beta = 15°$, $\gamma = 125°$.
13. $a = 21$, $b = 35$, $\gamma = 110°$.
14. $b = 17$, $c = 19$, $\alpha = 115°$.
15. $a = 12$, $b = 14$, $c = 22$.
16. $a = 1.5$, $b = 3$, $c = 2$.

17. Prove that the area of any triangle is equal to $\frac{1}{2}ab\sin\gamma$, or $\frac{1}{2}ac\sin\beta$, or $\frac{1}{2}bc\sin\alpha$.

18. Derive the Law of Sines by circumscribing a circle about a triangle ABC and showing that $2R = \frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma}$, where R is the radius of the circle.

HINT. In Fig. 99, $\angle D = \alpha$. (Why?)

$$\angle CBD = 90°.$$

Using the right triangle CBD will lead to the required result. Or use Fig. 100, proving that $\angle BOD = \alpha$ and then using the right triangle BOD to establish the result.

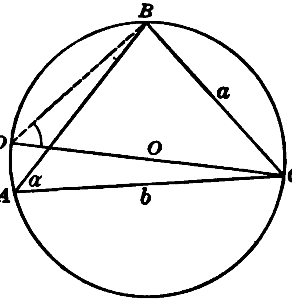


FIG. 99

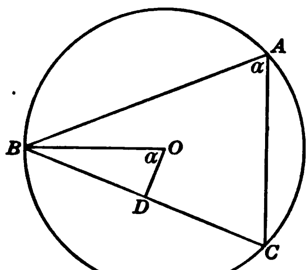


FIG. 100

19. Use the Law of the Projections to derive the Law of Cosines *algebraically*.

HINT. Multiply the equations (7), (8), and (9) by a, b, and c respectively, and combine the three resulting equations by addition and subtraction.

123. In general three independent data are sufficient to determine, and hence to solve, any triangle (but observe that the three angles are not three independent data). There are four possible combinations of sides and angles that are essentially different, and it may be found convenient to regard them as four "cases," or groups of data, for the solution of a triangle. The four are as follows:

CASE I. Given two angles and one side.

CASE II. Given two sides and the included angle.

CASE III. Given the three sides.

CASE IV. Given two sides and the angle opposite one of them.

It will be noted that examples of each of these possible combinations have occurred in the exercises on page 158. Before reading farther the student should make constructions of triangles from given parts, according to each of the first three cases. In each case, at least one example of an acute-angled triangle and one of an obtuse-angled triangle should be taken.

124. The "ambiguous case" in solution of a triangle. In Case IV appears a slight difference from the other three cases. The construction is, to be sure, equally simple. For instance, if we are given a, b, and α, we construct first the given angle α and lay off on one side the length $AC = b$; then, from C as center, with radius equal to the given opposite side a, we describe an arc which may cut the side AX in two points B_1 and B_2. Then *either* of the triangles ACB_1 or ACB_2 is a correct solution, for either contains the two given sides and the angle α opposite the side a.

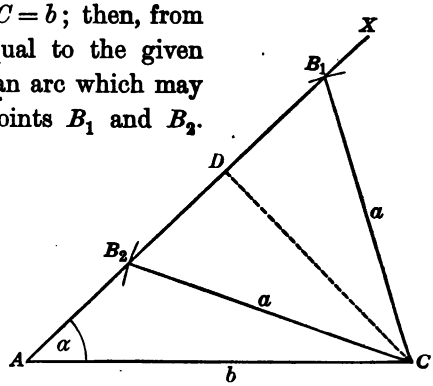


FIG. 101

Case IV is accordingly often called the "ambiguous case" in the construction and computation of triangles, because, when the sides and angles are as in Fig. 101, there are two equally correct solutions or constructions.

125. This ambiguity may, however, not occur if the relations of the given sides and angles are different from what they were in the figure above. If, for example, the side a is equal to or greater than the side b, there will be only one intersection with the line AX, and hence only one triangle can be constructed. Or, again, the side a may be exactly equal to the perpendicular distance CD from C to the opposite side AX, and in this case the arc with radius a will be tangent to AX, thus determining but one point BD. The *right* triangle ADC is then the only construction. Note that in case this happens, $a = CD = b \sin \alpha$. Finally, the side a may be shorter than the perpendicular distance from

C to AX, and then the arc with radius a will not meet AX at all, so that *no* construction is possible. In this case $a < b \sin \alpha$. A careful figure should be drawn for each of these possibilities.

126. Summarizing, in Case IV, when a, b, and α are given, α being an acute angle, there will be *two* solutions when and only when a is less than b and at the same time greater than $b \sin \alpha$. If $a \geqq b$ there will be only one solution; if $a = b \sin \alpha$ there will also be one solution, in this case a right triangle; and, finally, if $a < b \sin \alpha$ there will be no solution.

EXERCISES

1. Show that in Case IV, if the given angle is obtuse, there cannot be two solutions. When will there be *none*?

2. Determine the number of possible constructions (solutions) in each of the following cases:

(1) $\alpha = 30°$, $b = 10$, $a = 12$.
(2) $\alpha = 30°$, $b = 5$, $a = 3$.
(3) $\alpha = 30°$, $b = 100$, $a = 50$.
(4) $\beta = 30°$, $b = 25$, $a = 50$.
(5) $\gamma = 30°$, $b = 25$, $c = 30$.
(6) $\beta = 30°$, $b = 15$, $c = 25$.
(7) $\beta = 60°$, $b = 7$, $a = 10$.
(8) $\gamma = 20°$, $a = 20$, $c = 10$.
(9) $\gamma = 110°$, $b = 50$, $c = 100$.
(10) $\gamma = 150°$, $c = 25$, $a = 30$.

3. In Fig. 101, writing c_1 for AB_1, and c_2 for AB_2, prove that $c_1 + c_2 = 2\,b \cdot \cos \alpha$.

4. Show that (in the same figure) $c_1 - c_2 = 2\,a \cdot \cos \beta_1$.

5. Solve the following triangles, checking the results both by measurement and by computation. In the two-solution case, check by the formula of Ex. 3.

(1) $a = 39$, $b = 65$, $\alpha = 25°$.
(2) $b = 23$, $c = 4.1$, $\beta = 31°$.
(3) $a = 16$, $c = 24$, $\gamma = 49°$.
(4) $b = 78$, $c = 50$, $\gamma = 62°$.
(5) $a = 153$, $b = 136$, $\beta = 40°$.

MISCELLANEOUS PROBLEMS

In the following list of miscellaneous problems, draw an accurate figure whenever possible, and in every case check the result by computation:

1. In order to find the distance between two objects A and B, separated by a swamp, a station C is chosen, and the distances $CA = 355$ ft., $CB = 418$ ft., and $\angle ACB = 36°$ are measured. Find the distance from A to B.

2. Two objects, A and B, were observed from a ship to be in a line bearing N. 15° E. The ship then sailed N.W. 5 miles, when it was found that A bore due east and B bore N.E. Find the distance from A to B.

3. In a circle with radius 3 find the area of the part included between two parallel chords (on the same side of the center) whose lengths are 4 and 5.

4. The angle of elevation of a tower is at one point 63° 30′; at a point 500 ft. farther from the tower, in a straight line, it is 32° 15′. Find the height of the tower.

5. A tower makes an angle of 113° 12′ with the hillside on which it stands; and at a distance of 89 ft. from the base, measured down the hill, the angle subtended by the tower is 23° 27′. Find the height of the tower.

6. To determine the distance between two points A and B, a base-line CD is measured, 500 ft. long, and the following angles are observed: $ACB = 58° 20′$, $ACD = 95° 20′$, $ADB = 53° 30′$, $BDC = 98° 45′$. Find the length AB.

7. Two inaccessible points A and B are visible from D, but no other point can be found from which both are visible. A point C is taken, from which A and D can be seen, and CD is found to be 200 ft., while $\angle ADC = 89°$ and $\angle ACD = 50° 30′$. Then a point E is taken, from which D and B are visible, and DE is found to be 200 ft., $\angle BDE = 54° 30′$, $\angle BED = 88° 30′$. At D, $\angle ADB$ is observed to be 72° 30′. Compute the distance AB.

8. A pole 10 ft. high stands vertically, and from its foot the angle of elevation of the top of a tree is 32° 27′. The angle of elevation of the top of the pole from the foot of the tree is 14° 48′. Find the distance between the tree and the pole, and the height of the tree.

127. The half-angle formulas. We now turn from the applications of the trigonometric functions to the study of further theorems concerning the functions themselves. Problems 5–8 on page 67 gave four important relations among the trigonometric functions of an angle. We shall now discover the relations that exist between the functions of an angle and those of an angle twice as large.

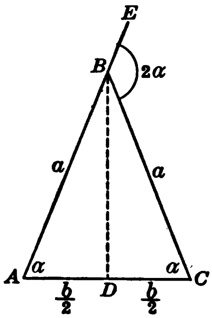


FIG. 102

128. Let ABC (Fig. 102) be an isosceles triangle, and let $\angle BAC = \angle BCA = \alpha$. Let each of the equal sides be represented by a, and the base by b. If AB be produced, the exterior angle $CBE = 2\alpha$. (Why?)

If we now apply the Law of Sines to the triangle ABC, we get

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\angle ABC} = \frac{b}{\sin(180° - 2\alpha)} = \frac{b}{\sin 2\alpha}. \qquad (1)$$

But $\qquad b = 2\,DC = 2\,a\cos\alpha, \qquad (2)$

since $\qquad DC = a\cos\alpha.$

Putting this value of b into equation (1),

$$\frac{a}{\sin\alpha} = \frac{2\,a\cos\alpha}{\sin 2\alpha};$$

that is, $\qquad \boldsymbol{\sin 2\alpha = 2\sin\alpha\cos\alpha.} \qquad (3)$

Next, let us apply the Law of Projections (p. 156) to the triangle ABC:

$$a = b\cos\alpha + a\cos(180° - 2\alpha) = 2\,a\cos^2\alpha - a\cos 2\alpha,$$

since $b = 2\,a\cos\alpha$, by (2); that is,

$$\boldsymbol{\cos 2\alpha = 2\cos^2\alpha - 1.} \qquad (4)$$

129. This relation can be written in other forms by making use of the fact that (Ex. 5, p. 67)

$$\sin^2\alpha + \cos^2\alpha = 1.$$

Therefore $\qquad \cos 2\alpha = 2(1 - \sin^2\alpha) - 1;$

that is, $\qquad \boldsymbol{\cos 2\alpha = 1 - 2\sin^2\alpha.} \qquad (5)$

Replacing the term 1 in equation (5) by $\cos^2\alpha + \sin^2\alpha$,

$$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha. \tag{6}$$

It will be noted that these formulas have been proved for any acute angle α; they are true for any angle whatever, but that fact must for the present be left without proof.

EXERCISES

1. Given $\sin 10° = .1736$, $\cos 10° = .9848$, find the values of $\sin 20°$, $\cos 20°$, and $\tan 20°$ to four decimal places.

2. Prove that $\tan 2\alpha = \dfrac{2\tan\alpha}{1-\tan^2\alpha}$.

3. In formulas (4) and (5), change 2α to α, which requires that α be changed to $\dfrac{\alpha}{2}$, thus getting

$$\cos\alpha = 2\cos^2\frac{\alpha}{2} - 1 \quad\text{and}\quad \cos\alpha = 1 - 2\sin^2\frac{\alpha}{2},$$

and thence derive the formulas

$$\cos\frac{\alpha}{2} = \sqrt{\frac{1+\cos\alpha}{2}} \quad\text{and}\quad \sin\frac{\alpha}{2} = \sqrt{\frac{1-\cos\alpha}{2}}.$$

4. Find the values of $\sin 15°$, $\cos 15°$, and $\tan 15°$ from the fact that $\cos 30° = \dfrac{\sqrt{3}}{2}$ (p. 68).

Ans. $\sin 15° = \dfrac{1}{2}\sqrt{2-\sqrt{3}} = \dfrac{\sqrt{6}-\sqrt{2}}{4}$,

$\cos 15° = \dfrac{1}{2}\sqrt{2+\sqrt{3}} = \dfrac{\sqrt{6}+\sqrt{2}}{4}$,

$\tan 15° = \sqrt{\dfrac{2-\sqrt{3}}{2+\sqrt{3}}} = 2-\sqrt{3}$.

5. Find the values of $\sin 22\frac{1}{2}°$, $\cos 22\frac{1}{2}°$, and $\tan 22\frac{1}{2}°$ from the known values of the functions of $45°$.

Ans. $\sin 22\frac{1}{2}° = \frac{1}{2}\sqrt{2-\sqrt{2}}$,

$\cos 22\frac{1}{2}° = \frac{1}{2}\sqrt{2+\sqrt{2}}$,

$\tan 22\frac{1}{2}° = \sqrt{\dfrac{2-\sqrt{2}}{2+\sqrt{2}}} = \sqrt{2}-1$.

6. Prove that $\tan\dfrac{\alpha}{2} = \sqrt{\dfrac{1-\cos\alpha}{1+\cos\alpha}} = \dfrac{\sin\alpha}{1+\cos\alpha} = \dfrac{1-\cos\alpha}{\sin\alpha}$.

7. Derive the values of the functions of 15° *geometrically* by constructing a regular dodecagon and computing the exact values of the ratios of the apothem[1] and side to the radius. Compare with the results of Ex. 4.

8. As in Ex. 7, construct a regular octagon and thereby find the exact values of the functions of $22\frac{1}{2}°$.

9. As in Ex. 7, use the regular decagon to find the values of the functions of 18° (cf. Ex. 8, p. 60).

$$\textit{Ans.}\quad \sin 18° = \frac{\sqrt{5}-1}{4},$$
$$\cos 18° = \tfrac{1}{4}\sqrt{10+2\sqrt{5}},$$
$$\tan 18° = \sqrt{\frac{6-2\sqrt{5}}{10+2\sqrt{5}}} = \frac{1}{5}\sqrt{25-10\sqrt{5}}.$$

10. As in Ex. 7, use the regular pentagon to find the values of the functions of 36°.

$$\textit{Ans.}\quad \sin 36° = \tfrac{1}{4}\sqrt{10-2\sqrt{5}},$$
$$\cos 36° = \frac{\sqrt{5}+1}{4},$$
$$\tan 36° = \sqrt{5-2\sqrt{5}}.$$

11. Prove that $\tan 7\frac{1}{2}° = \sqrt{6} - \sqrt{3} + \sqrt{2} - 2$.

130. The problems which follow are designed to aid the student in fixing in mind the relations among the trigonometric functions which we have thus far considered, and also to develop the power of discovering new relations. The illustrative examples should be studied, as indicating the general method to be followed, the details, however, varying from problem to problem.

Example 1. Prove that $\cos^4\theta - \sin^4\theta = \cos 2\,\theta$.

Proof.
$$\cos^4\theta - \sin^4\theta = (\cos^2\theta - \sin^2\theta)(\cos^2\theta + \sin^2\theta)$$
$$= \cos^2\theta - \sin^2\theta,$$

since
$$\cos^2\theta + \sin^2\theta = 1.$$

But
$$\cos^2\theta - \sin^2\theta = \cos 2\,\theta;$$

hence the theorem is proved.

[1] Apothem: the distance from the center to a side of a regular polygon.

Example 2. Prove that $\dfrac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}} = \sin x.$

Proof. $$1+\tan^2\frac{x}{2} = \sec^2\frac{x}{2}.$$

Therefore $$\frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}} = \frac{2\tan\frac{x}{2}}{\sec^2\frac{x}{2}} = 2\tan\frac{x}{2}\cdot\cos^2\frac{x}{2}.$$

Now $$\tan\frac{x}{2} = \frac{\sin\frac{x}{2}}{\cos\frac{x}{2}}.$$

Therefore $$2\tan\frac{x}{2}\cdot\cos^2\frac{x}{2} = 2\sin\frac{x}{2}\cos\frac{x}{2} = \sin x.$$ Q.E.D.

Another method. This formula can also be proved as follows:

$$\tan\frac{x}{2} = \frac{\sin x}{1+\cos x}$$

and $$\tan^2\frac{x}{2} = \frac{1-\cos x}{1+\cos x}.$$

Therefore $$1+\tan^2\frac{x}{2} = 1+\frac{1-\cos x}{1+\cos x} = \frac{2}{1+\cos x}.$$

Therefore $$\frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}} = \frac{\frac{2\sin x}{1+\cos x}}{\frac{2}{1+\cos x}} = \sin x.$$ Q.E.D.

EXERCISES

Prove each of the following formulas:

1. $\tan x + \cot x = 2\csc 2x.$

2. $\sin\frac{\theta}{2} + \cos\frac{\theta}{2} = \sqrt{1+\sin\theta}.$

3. $\tan\left(45^\circ + \frac{x}{2}\right) = \sec x + \tan x.$

Hint. $\tan\left(45^\circ + \frac{x}{2}\right) = \tan\frac{90^\circ + x}{2}.$

4. $\cot x - \tan x = 2\cot 2x.$

5. $\dfrac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}} = \cos x.$

6. $\dfrac{2\sin^3 x}{1-\cos x} = 2\sin x + \sin 2x.$

7. $\sin\frac{x}{2} - \cos\frac{x}{2} = \sqrt{1 - \sin x}.$

8. $\sin 2x \sin x = (1 - \cos 2x)\cos x.$

9. $\tan\frac{\theta}{2} = \frac{1 + \sin\theta - \cos\theta}{1 + \sin\theta + \cos\theta}.$

10. $\cos^4 x + \sin^4 x = 1 - \frac{1}{2}\sin^2 2x.$

11. $\csc x - 2\cos x \cot 2x = 2\sin x.$

12. $\cos^6 x - \sin^6 x = \cos 2x\left(1 - \frac{\sin^2 2x}{4}\right).$

13. $\cos^6 x + \sin^6 x = 1 - 3\sin^2 x \cos^2 x.$

14. $\sin 4x = 4\sin x \cos x - 8\sin^3 x \cos x = 8\cos^3 x \sin x - 4\cos x \sin x.$

15. $\cos 4x = 1 - 8\cos^2 x + 8\cos^4 x = 1 - 8\sin^2 x + 8\sin^4 x.$

16. $\tan 2x + \sec 2x = \frac{\cos x + \sin x}{\cos x - \sin x}.$

17. $\sec 2x = \frac{\sec^2 x}{2 - \sec^2 x}.$

18. $\tan\theta \cdot \tan 2\theta = \sec 2\theta - 1.$

19. $\tan^2\frac{\theta}{2} = \frac{2\sin\theta - \sin 2\theta}{2\sin\theta + \sin 2\theta}.$

20. $1 + \sec 2\theta + \tan 2\theta = \frac{2}{1 - \tan\theta}.$

21. $\frac{\sin\alpha + \sin 2\alpha}{\cos\alpha - \cos 2\alpha} = \cot\frac{\alpha}{2}.$

22. $\left(1 + \cot^2\frac{\alpha}{2}\right)\sin\alpha \cdot \tan\frac{\alpha}{2} = 2.$

23. $\tan 2\alpha = \frac{2\cot\alpha}{\cot^2\alpha - 1}.$

24. $\sin^2\frac{x}{2}\left(\cot\frac{x}{2} - 1\right)^2 = 1 - \sin x.$

25. $\frac{\cos x - \sin x}{\cos x + \sin x} = \frac{1 - \sin 2x}{\cos 2x} = \frac{\cos 2x}{1 + \sin 2x}.$

26. $\tan 2\alpha = \frac{\tan\alpha}{1 - \tan\alpha} + \frac{\tan\alpha}{1 + \tan\alpha} = \frac{1}{1 - \tan\alpha} - \frac{1}{1 + \tan\alpha}.$

27. $1 + \sin 2\theta = \frac{(1 + \tan\theta)^2}{1 + \tan^2\theta} = \frac{(1 + \cot\theta)^2}{1 + \cot^2\theta} = \frac{(1 + \tan\theta)(1 + \cot\theta)}{\tan\theta + \cot\theta}.$

28. $\dfrac{\sin 2x}{1+\sin 2x}=\dfrac{2\tan x}{(1+\tan x)^2}=\dfrac{2\cot x}{(1+\cot x)^2}=\dfrac{2}{(1+\tan x)(1+\cot x)}.$

29. $2\cos\dfrac{45^\circ}{2^n}=\sqrt{2+\sqrt{2+\sqrt{2+\cdots+\sqrt{2}}}}$ ($n+1$ radical signs).

30. $2\cos\dfrac{30^\circ}{2^n}=\sqrt{2+\sqrt{2+\cdots+\sqrt{3}}}$ ($n+1$ radical signs).

131. Functions of the sum or difference of two angles. For the further study of the trigonometric functions no theorems are more important than those now to be proved, which enable us to find any function of the sum or difference of two angles from the functions of each angle. Two proofs will be given, one algebraic and the other geometric.

132. First proof. This proof uses the Law of Sines, and also the formula (7) on page 156: $c = a\cos\beta + b\cos\alpha$. By the Law of Sines (in the form given in Ex. 18, p. 159), $c = 2R\sin\gamma$, $b = 2R\sin\beta$, and $a = 2R\sin\alpha$.

Therefore $\quad 2R\sin\gamma = 2R\sin\alpha\cos\beta + 2R\sin\beta\cos\alpha.$

Now, since $\quad \alpha+\beta+\gamma = 180^\circ,$

$$\gamma = 180^\circ - (\alpha+\beta);$$

therefore $\quad \sin\gamma = \sin[180^\circ - (\alpha+\beta)] = \sin(\alpha+\beta).$

Therefore $\quad \boldsymbol{\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta}.$ (I)

To find the value of $\cos(\alpha+\beta)$: Begin with equations (8) and (9) on page 156,

$$a = b\cos\gamma + c\cos\beta,$$
$$b = a\cos\gamma + c\cos\alpha.$$

Multiplying these together, we get

$$ab = ab\cos^2\gamma + c\cos\gamma\,(b\cos\alpha + a\cos\beta) + c^2\cos\alpha\cos\beta.$$

But, by (7) on page 156, $b\cos\alpha + a\cos\beta = c$.

Therefore

$$ab(1-\cos^2\gamma) = c^2\cos\gamma + c^2\cos\alpha\cos\beta = c^2(\cos\gamma + \cos\alpha\cos\beta).$$

By the Law of Sines, as above, $a = 2R\sin\alpha$, $b = 2R\sin\beta$, and $c = 2R\sin\gamma$; also $1-\cos^2\gamma = \sin^2\gamma$.

Therefore

$$4R^2\sin\alpha\sin\beta\cdot\sin^2\gamma = 4R^2\sin^2\gamma(\cos\gamma + \cos\alpha\cos\beta).$$

Dividing both sides of this equation by $4R^2\sin^2\gamma$,

$$\sin\alpha\sin\beta = \cos\gamma + \cos\alpha\cos\beta.$$

And, since $\alpha + \beta + \gamma = 180°$,

$$\cos\gamma = \cos[180° - (\alpha+\beta)] = -\cos(\alpha+\beta).$$

Therefore $\qquad \boldsymbol{\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta.} \qquad$ (II)

133. Second proof. Let α and β be any two acute angles. Place them with a common vertex O and a common side ON, as in Fig. 103, so that the angle $MOP = \alpha + \beta$. From any point A in OM draw AB perpendicular to ON at B, and produce it to meet OP at C. Notice that this assumes that $\alpha + \beta$ is an acute angle. Then we can express the areas of the triangles formed, as follows (Ex. 17, p. 159):

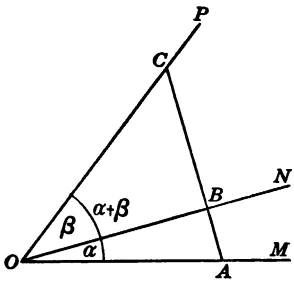


FIG. 103

Area $\triangle AOC = \frac{1}{2} OA \cdot OC \cdot \sin(\alpha+\beta)$.

Area $\triangle AOB = \frac{1}{2} OA \cdot OB \cdot \sin\alpha$.

Area $\triangle BOC = \frac{1}{2} OB \cdot OC \cdot \sin\beta$.

Since $\triangle AOC = \triangle AOB + \triangle BOC$,

therefore

$$\tfrac{1}{2} OA \cdot OC \cdot \sin(\alpha+\beta) = \tfrac{1}{2} OA \cdot OB \cdot \sin\alpha + \tfrac{1}{2} OB \cdot OC \cdot \sin\beta.$$

Therefore

$$\sin(\alpha+\beta) = \frac{OB}{OC}\sin\alpha + \frac{OB}{OA}\sin\beta.$$

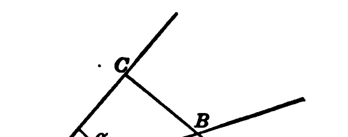


FIG. 104

But $\qquad \dfrac{OB}{OC} = \cos\beta$

and $\qquad \dfrac{OB}{OA} = \cos\alpha.$

Therefore $\qquad \boldsymbol{\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta.} \qquad$ (I)

This same method can be used to find the value of $\sin(\alpha - \beta)$, as follows:

Let $\angle AOC = \alpha$ (Fig. 104) and $\angle BOC = \beta$, AC being perpendicular to OC. Then $\angle AOB = \alpha - \beta$, and, just as before,

$$\tfrac{1}{2} OA \cdot OC \cdot \sin\alpha = \tfrac{1}{2} OA \cdot OB \cdot \sin(\alpha-\beta) + \tfrac{1}{2} OB \cdot OC \cdot \sin\beta.$$

Therefore $$\sin(\alpha-\beta)=\frac{OC}{OB}\sin\alpha-\frac{OC}{OA}\sin\beta,$$

or $$\mathbf{\sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta.} \qquad \text{(III)}$$

From (I) and (III) the values of $\cos(\alpha+\beta)$ and $\cos(\alpha-\beta)$ can be found. For
$$\cos\theta=\sin(90^\circ-\theta).$$

Therefore $$\begin{aligned}\cos(\alpha+\beta)&=\sin[90^\circ-(\alpha+\beta)]\\&=\sin[(90^\circ-\alpha)-\beta].\end{aligned}$$

This last expression is in the form of the sine of the difference of two angles, both of which are acute, since $90^\circ-\alpha$ is acute if α is. Therefore we can apply (III), getting
$$\begin{aligned}\sin[(90^\circ-\alpha)-\beta]&=\sin(90^\circ-\alpha)\cos\beta-\cos(90^\circ-\alpha)\sin\beta\\&=\cos\alpha\cos\beta-\sin\alpha\sin\beta;\end{aligned}$$

that is, $$\mathbf{\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta.} \qquad \text{(II)}$$

To obtain the value of $\cos(\alpha-\beta)$:
$$\cos(\alpha-\beta)=\sin[90^\circ-(\alpha-\beta)]=\sin[(90^\circ-\alpha)+\beta].$$

Since the angles $(90^\circ-\alpha)$ and β are both acute, we can apply (I):
$$\begin{aligned}\sin[(90^\circ-\alpha)+\beta]&=\sin(90^\circ-\alpha)\cos\beta+\cos(90^\circ-\alpha)\sin\beta\\&=\cos\alpha\cos\beta+\sin\alpha\sin\beta.\end{aligned}$$

Therefore $$\mathbf{\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta.} \qquad \text{(IV)}$$

134. The first method of proof (§ 132) assumes that α and β can be taken as two of the angles of a triangle; that is, that their sum is less than 180°. The second proof (§ 133) assumes that α, β, and $\alpha+\beta$ are acute. We can, however, remove these restrictions and establish the fact that formulas I–IV hold *for any angles whatever.*

To accomplish this, take $\alpha=90^\circ+\alpha'$, where α' is an acute angle, and suppose β to be acute, as before. Then
$$\sin(\alpha+\beta)=\sin[90^\circ+\alpha'+\beta]=\cos(\alpha'+\beta).$$

Since α' and β are acute angles, (II) can be applied, giving
$$\cos(\alpha'+\beta)=\cos\alpha'\cos\beta-\sin\alpha'\sin\beta.$$

But $$\cos \alpha' = \cos (\alpha - 90°) = \sin \alpha,$$
and $$\sin \alpha' = \sin (\alpha - 90°) = - \cos \alpha.$$
Therefore $$\cos (\alpha' + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta.$$
Therefore $$\sin (\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta.$$

Hence formula (I) is true if one angle is obtuse and the other acute, and by exactly the same process we can establish its truth if we add 90° to any angle α' for which the formula has already been proved. Thus, we can add 90° to β, proving the formula true when *both* angles are obtuse; then we can add 90° to α, proving its truth when one angle is between 180° and 270°, the other being either acute or obtuse; and so on. The same reasoning holds for successive *subtractions* of 90°. Hence (I) is true for angles of *any magnitude whatever*. The theorem is, as was said before, of the very greatest importance. It is called the *Addition Theorem* for the trigonometric functions.

Similar reasoning can be used to establish the truth of II, III, and IV for angles of any magnitude.

EXERCISES

1. Work through the same reasoning as above to prove (II) in general.

2. Prove (III) by changing β to $-\beta$ in (I). (This is allowable, since (I) has been proved true for all angles.)

3. Prove (IV) in a similar way.

4. Find the values of $\sin 75°$ and $\cos 75°$ from the known values of the functions of 45° and 30°.

5. Find the values of $\sin 15°$ and $\cos 15°$ from the values of the functions of 45° and 30°. Compare with the results of Ex. 4 and also of Ex. 4, p. 164.

6. Derive formulas for $\tan(\alpha + \beta)$ and $\tan(\alpha - \beta)$ by using I–IV.

Ans. $$\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta};\quad \tan(\alpha-\beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}.$$

7. Prove that $$\cot(\alpha + \beta) = \frac{\cot\alpha\cot\beta - 1}{\cot\beta + \cot\alpha},$$

and that $$\cot(\alpha - \beta) = \frac{\cot\alpha\cot\beta + 1}{\cot\beta - \cot\alpha}.$$

8. Prove formulas (I) and (II) by using Fig. 105. Show that

$$\sin(\alpha+\beta)=\frac{DE+FB}{OB}$$

$$=\frac{DE}{OE}\cdot\frac{OE}{OB}+\frac{FB}{EB}\cdot\frac{EB}{OB}$$

$$=\sin\alpha\cos\beta+\cos\alpha\sin\beta;$$

and, similarly, derive (II) by starting with

$$\cos(\alpha+\beta)=\frac{OC}{OB}=\frac{OD-FE}{OB}.$$

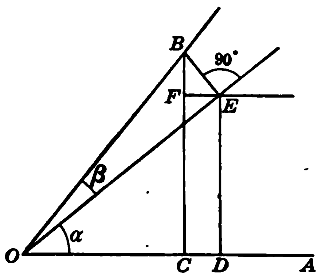


Fig. 105

9. In (I) and (II) let $\beta=\alpha$, thus getting formulas for $\sin 2\alpha$ and $\cos 2\alpha$. Compare with the results of §§ 127, 128. Notice that we are now for the first time able to assert that these results are true for all angles whatsoever. The "half-angle formulas" of Ex. 3, p. 164, are thereby also assured universal validity, because they were derived by purely algebraic processes from the values for $\cos 2\alpha$.

10. Find the values of the functions of 75° from those of 150°, and compare with the results of Ex. 4 above.

11. Prove that

$$\sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta$$

by using Fig. 106. ABC is any triangle, CD is the altitude from C, $DE=AD$; show that $\angle ECB=\alpha-\beta$, and then apply the Law of Sines to the triangle CBE.

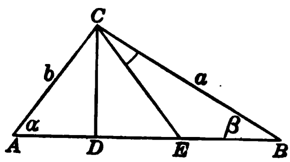


Fig. 106

12. Prove the Addition Theorem from the Law of Sines, using the fact that if $\frac{a}{b}=\frac{c}{d}$, then $\frac{a+c}{b+d}=\frac{a}{b}$.

Prove each of the following formulas (13–30):

13. $\dfrac{\sin(x+y)}{\cos x\cos y}=\tan x+\tan y.$

14. $\tan(x+y)=\dfrac{\sin 2x+\sin 2y}{\cos 2x+\cos 2y}.$

15. $\cos(x+30^\circ)+\cos(x-30^\circ)=\sqrt{3}\cdot\cos x.$

16. $\sin(x+60^\circ)+\sin(x-60^\circ)=\sin x.$

17. $\sin\theta+\sin(\theta-120^\circ)+\sin(60^\circ-\theta)=0.$

18. $\sin 3x = 3\sin x - 4\sin^3 x$.

19. $\cos 3x = 4\cos^3 x - 3\cos x$.

20. $\sec(\alpha + 45°)\sec(\alpha - 45°) = 2\sec 2\alpha$.

21. $\tan(45° + \alpha) - \tan(45° - \alpha) = 2\tan 2\alpha$.

22. $\tan(\theta + 45°) + \tan(45° - \theta) = 2\sec 2\theta$.

23. $\sin(\alpha + \beta) + \sin(\alpha - \beta) = 2\sin\alpha\cos\beta$.

24. $\sin(\alpha + \beta) - \sin(\alpha - \beta) = 2\cos\alpha\sin\beta$.

25. $\cos(\alpha + \beta) + \cos(\alpha - \beta) = 2\cos\alpha\cos\beta$.

26. $\cos(\alpha + \beta) - \cos(\alpha - \beta) = -2\sin\alpha\sin\beta$.

27. $\sin(\alpha + \beta)\sin(\alpha - \beta) = \sin^2\alpha - \sin^2\beta$.

28. $\cos(\alpha + \beta)\cos(\alpha - \beta) = \cos^2\alpha + \cos^2\beta - 1 = \cos^2\alpha - \sin^2\beta$
$= \cos^2\beta - \sin^2\alpha$.

29. $\sin(\alpha + \beta)\cos(\alpha - \beta) = \sin\alpha\cos\alpha + \sin\beta\cos\beta$.

30. $\sin(\alpha - \beta)\cos(\alpha + \beta) = \sin\alpha\cos\alpha - \sin\beta\cos\beta$.

31. Obtain the values of the functions of 18° by the following method: $\sin 36° = 2\sin 18°\cos 18°$, and also $\sin 36° = \cos 54°$. But, by Ex. 19, above, $\cos 54° = 4\cos^3 18° - 3\cos 18°$. Therefore $2\sin 18°\cos 18° = 4\cos^3 18° - 3\cos 18°$. Solve this for $\sin 18°$ and compare the result with that of Ex. 9, p. 165.

32. Prove that $\sin(\alpha + \beta)\cos\beta - \cos(\alpha + \beta)\sin\beta = \sin\alpha$.

Solution. This can be proved by writing out the values of $\sin(\alpha + \beta)$ and $\cos(\alpha + \beta)$, but it is simpler to observe carefully the combination of terms that we have here, namely, $\sin x\cos\beta - \cos x\sin\beta$, x representing $\alpha + \beta$. This is the value of $\sin(x - \beta)$, that is, of $\sin(\alpha + \beta - \beta)$, or $\sin\alpha$.

33. Prove: $\sin(\alpha - \beta)\cos\beta + \cos(\alpha - \beta)\sin\beta = \sin\alpha$.

34. Prove: $\sin(\alpha + \beta)\sin\beta + \cos(\alpha + \beta)\cos\beta = \cos\alpha$.

35. Prove: $\dfrac{\tan(\alpha - \beta) + \tan\beta}{1 - \tan(\alpha - \beta)\tan\beta} = \tan\alpha$.

36. Prove: $\dfrac{\cos\alpha - \cos\theta\cos(\theta + \alpha)}{\cos\alpha\sin(\theta + \alpha)} = \sin\theta\sec\alpha$.

37. Prove that the angle θ, made by a line whose slope is m_1 with a line whose slope is m_2, is given by $\tan\theta = \dfrac{m_1 - m_2}{1 + m_1 m_2}$.

38. Find the angle that the line $3x - 2y + 1 = 0$ makes with the line $x + y - 3 = 0$.

39. Find the angles of the triangle whose vertices are the points (3, 5), (7, −1), and (−2, 3).

40. The theorem known as the Ptolemaic Theorem[1] is as follows: In any inscribed quadrilateral the sum of the products of the opposite sides is equal to the product of the diagonals.

(a) Prove this theorem by elementary geometry.

(b) Assuming its truth, prove the Addition Theorem.

HINTS. For (a) draw from B a line BE (not shown in Fig. 107), so that $\angle ABE = \beta$, E being the point where this line meets AC; then the triangles ABE and BDC are similar, as are the triangles BEC and ABD. From these facts the theorem can be proved.

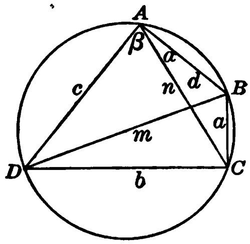


FIG. 107

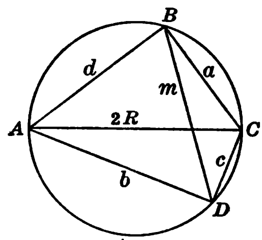


FIG. 108

For (b) construct the given angles α and β on opposite sides of the diameter through A (Fig. 108), and complete the inscribed quadrilateral $ABCD$. Then apply the fact that $a = 2R\sin\alpha$ etc. to the right triangles ABC and ACD; also the Law of Sines to the triangle ABD.

135. Sum and difference of two sines or two cosines. The equations that were proved in Exs. 23–26, p. 173, were as follows:

$$\sin(\alpha+\beta) + \sin(\alpha-\beta) = 2\sin\alpha\cos\beta, \tag{1}$$

$$\sin(\alpha+\beta) - \sin(\alpha-\beta) = 2\cos\alpha\sin\beta, \tag{2}$$

$$\cos(\alpha+\beta) + \cos(\alpha-\beta) = 2\cos\alpha\cos\beta, \tag{3}$$

$$\cos(\alpha+\beta) - \cos(\alpha-\beta) = -2\sin\alpha\sin\beta. \tag{4}$$

These equations give the sum or difference of two sines, or of two cosines, in the form of a product, a form which we shall often

[1] From Ptolemy, the famous Greek astronomer of the second century. This theorem is included in his great work called the "Almagest," which explained the astronomic system which, under the name of the "Ptolemaic system" was universally accepted for 1400 years. It was only overthrown after a long struggle between its adherents and those of the system of Copernicus. See the articles "Ptolemy" and "Copernicus" in the Encyclopedia Britannica.

find useful. These equations are written in a more practical form by setting $\alpha + \beta = x$ and $\alpha - \beta = y$.

Therefore $\quad 2\alpha = x + y \quad$ and $\quad 2\beta = x - y$.

$$\alpha = \frac{x+y}{2} \quad \text{and} \quad \beta = \frac{x-y}{2}.$$

Thus the equations (1) – (4) are equivalent to

$$\sin x + \sin y = 2 \sin \frac{x+y}{2} \cos \frac{x-y}{2}, \tag{5}$$

$$\sin x - \sin y = 2 \cos \frac{x+y}{2} \sin \frac{x-y}{2}, \tag{6}$$

$$\cos x + \cos y = 2 \cos \frac{x+y}{2} \cos \frac{x-y}{2}, \tag{7}$$

$$\cos x - \cos y = -2 \sin \frac{x+y}{2} \sin \frac{x-y}{2}. \tag{8}$$

EXERCISES

1. Prove that $\dfrac{\sin 33^\circ + \sin 3^\circ}{\cos 33^\circ + \cos 3^\circ} = \tan 18^\circ$.

Proof. By (5) and (7),

$$\frac{\sin 33^\circ + \sin 3^\circ}{\cos 33^\circ + \cos 3^\circ} = \frac{2 \sin \frac{36^\circ}{2} \cos \frac{30^\circ}{2}}{2 \cos \frac{36^\circ}{2} \cos \frac{30^\circ}{2}} = \frac{\sin 18^\circ}{\cos 18^\circ} = \tan 18^\circ.$$

2. Prove: $\dfrac{\sin 3\alpha + \sin 5\alpha}{\cos 3\alpha + \cos 5\alpha} = \tan 4\alpha$.

3. Prove: $\dfrac{\sin 75^\circ + \sin 15^\circ}{\sin 75^\circ - \sin 15^\circ} = \tan 60^\circ$.

4. Prove: $\dfrac{\tan x + \tan y}{\cot x + \cot y} = \tan x \tan y$.

5. Prove: $\dfrac{\sin x + \sin y}{\sin x - \sin y} = \dfrac{\tan \frac{x+y}{2}}{\tan \frac{x-y}{2}}$.

6. Prove the formula (3), p. 174, by means of Fig. 109. This method of proof is found in an astronomical work of the tenth century, by an Arabian scholar, Ibn Jûnos. The formulas (1), (2), and (4) can also be proved in a similar way.

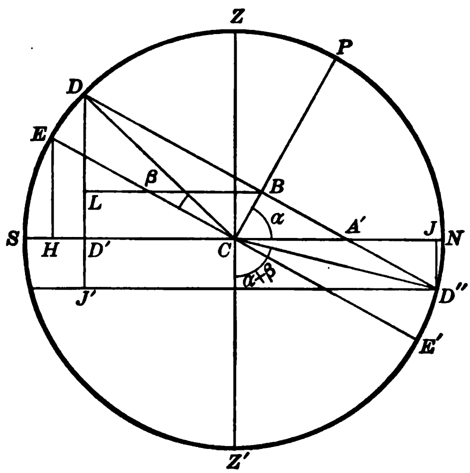


FIG. 109

HINTS. Letting $\angle NCP = \angle Z'CE' = \alpha,$

and $\angle E'CD'' = \angle ECD = \beta,$

it follows that $JD'' = \cos(\alpha + \beta)$ (if $r = 1$)

and $D'D = \cos(\alpha - \beta).$

Therefore $J'D = \cos(\alpha + \beta) + \cos(\alpha - \beta).$

Next, show $LD = \frac{1}{2} J'D = \cos\alpha \cdot \cos\beta$ by showing that $LD = BD \cdot \cos\alpha.$

7. $\cos x + \cos 3x + \cos 5x + \cos 7x = 4\cos x \cdot \cos 2x \cdot \cos 4x.$

Proof.

$$\cos x + \cos 3x = 2\cos 2x \cos x,$$
$$\cos 5x + \cos 7x = 2\cos 6x \cos x.$$

Therefore $\cos x + \cos 3x + \cos 5x + \cos 7x = 2\cos x(\cos 6x + \cos 2x)$

$$= 2\cos x(2\cos 4x\cos 2x)$$
$$= 4\cos x(\cos 2x\cos 4x). \quad \text{Q.E.D.}$$

8. $\sin x + \sin 3x + \sin 5x + \sin 7x = 4\cos x \cdot \cos 2x \cdot \sin 4x.$

9. $\dfrac{\tan\alpha + \tan\beta}{\tan\alpha - \tan\beta} = \dfrac{\sin(\alpha + \beta)}{\sin(\alpha - \beta)}.$

10. $\dfrac{\sin\alpha+\sin 2\alpha+\sin 3\alpha}{\cos\alpha+\cos 2\alpha+\cos 3\alpha}=\tan 2\alpha.$

11. $\dfrac{\sin\alpha+\sin 2\alpha+\sin 3\alpha+\sin 4\alpha}{\cos\alpha+\cos 2\alpha+\cos 3\alpha+\cos 4\alpha}=\tan\dfrac{5\alpha}{2}.$

If α, β, and γ are the angles of a triangle, prove that each of the following relations is true:

12. $\sin\alpha+\sin\beta+\sin\gamma=4\cos\dfrac{\alpha}{2}\cdot\cos\dfrac{\beta}{2}\cdot\cos\dfrac{\gamma}{2}.$

Proof. $$\sin\alpha+\sin\beta=2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}.$$

$$\sin\gamma=\sin[180^\circ-(\alpha+\beta)]=\sin(\alpha+\beta)$$

$$=2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha+\beta}{2}. \qquad ((3),\ \text{p. } 163)$$

Therefore $$\sin\alpha+\sin\beta+\sin\gamma=2\sin\frac{\alpha+\beta}{2}\left[\cos\frac{\alpha+\beta}{2}+\cos\frac{\alpha-\beta}{2}\right]$$

$$=2\sin\frac{180^\circ-\gamma}{2}\left[2\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\right]$$

$$=4\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2}. \qquad \text{Q.E.D.}$$

13. $\cos\alpha+\cos\beta+\cos\gamma=1+4\sin\dfrac{\alpha}{2}\sin\dfrac{\beta}{2}\sin\dfrac{\gamma}{2}.$

14. $\tan\alpha+\tan\beta+\tan\gamma=\tan\alpha\tan\beta\tan\gamma.$

15. $\sin\alpha+\sin\beta-\sin\gamma=4\sin\dfrac{\alpha}{2}\sin\dfrac{\beta}{2}\cos\dfrac{\gamma}{2}.$

16. $\cos\alpha+\cos\beta-\cos\gamma=-1+4\cos\dfrac{\alpha}{2}\cos\dfrac{\beta}{2}\sin\dfrac{\gamma}{2}.$

17. $\cot\dfrac{\alpha}{2}+\cot\dfrac{\beta}{2}+\cot\dfrac{\gamma}{2}=\cot\dfrac{\alpha}{2}\cot\dfrac{\beta}{2}\cot\dfrac{\gamma}{2}.$

18. $\tan\dfrac{\alpha}{2}\tan\dfrac{\beta}{2}+\tan\dfrac{\beta}{2}\tan\dfrac{\gamma}{2}+\tan\dfrac{\gamma}{2}\tan\dfrac{\alpha}{2}=1.$

19. $\cos^2\alpha+\cos^2\beta+\cos^2\gamma=1-2\cos\alpha\cos\beta\cos\gamma.$

20. $\sin^2\alpha+\sin^2\beta+\sin^2\gamma=2(1+\cos\alpha\cos\beta\cos\gamma).$

21. $1-\sin^2\dfrac{\alpha}{2}-\sin^2\dfrac{\beta}{2}-\sin^2\dfrac{\gamma}{2}=2\sin\dfrac{\alpha}{2}\sin\dfrac{\beta}{2}\sin\dfrac{\gamma}{2}.$

22. $\sin 2\alpha+\sin 2\beta+\sin 2\gamma=4\sin\alpha\sin\beta\sin\gamma.$

23. $\sin 2\alpha+\sin 2\beta-\sin 2\gamma=4\cos\alpha\cos\beta\sin\gamma.$

24. $\dfrac{\sin 2\alpha + \sin 2\beta + \sin 2\gamma}{\sin\alpha + \sin\beta + \sin\gamma} = 8 \sin\dfrac{\alpha}{2}\sin\dfrac{\beta}{2}\sin\dfrac{\gamma}{2}.$

25. $\cos 2\alpha + \cos 2\beta + \cos 2\gamma = -1 - 4\cos\alpha\cos\beta\cos\gamma.$

26. $\cos 2\alpha + \cos 2\beta - \cos 2\gamma = 1 - 4\sin\alpha\sin\beta\cos\gamma.$

27. $\cot\alpha + \cot\beta + \cot\gamma = \cot\alpha\cot\beta\cot\gamma + \csc\alpha\csc\beta\csc\gamma.$

28. $\sin^2\dfrac{\gamma}{2} = \dfrac{(\sin\beta + \sin\gamma - \sin\alpha)(\sin\gamma + \sin\alpha - \sin\beta)}{4\sin\alpha\sin\beta}.$

29. $\cos\dfrac{\alpha}{2} + \cos\dfrac{\beta}{2} + \cos\dfrac{\gamma}{2} = 4\cos\dfrac{\beta+\gamma}{4}\cos\dfrac{\alpha+\gamma}{4}\cos\dfrac{\alpha+\beta}{4}.$

30. $\sin 3\alpha + \sin 3\beta + \sin 3\gamma = -4\cos\dfrac{3\alpha}{2}\cos\dfrac{3\beta}{2}\cos\dfrac{3\gamma}{2}.$

31. $\cos^2\dfrac{\gamma}{2} = \dfrac{(\sin\alpha + \sin\beta - \sin\gamma)(\sin\alpha + \sin\beta + \sin\gamma)}{4\sin\alpha\sin\beta}.$

Changes in the Trigonometric Functions as the Angle changes

136. Our experience with the use of the trigonometric functions has given us considerable information as to the way in which they change as the angle changes. Let us now follow out these variations more carefully.

137. The sine. Since the sine of an angle equals the quotient of the ordinate by the radius vector, we shall find it easy to follow the changes in the sine function if we choose points on the terminal line of the angle such that the radius vector is constant. This will be accomplished if the points are taken on the circumference of a circle, as in Fig. 110.

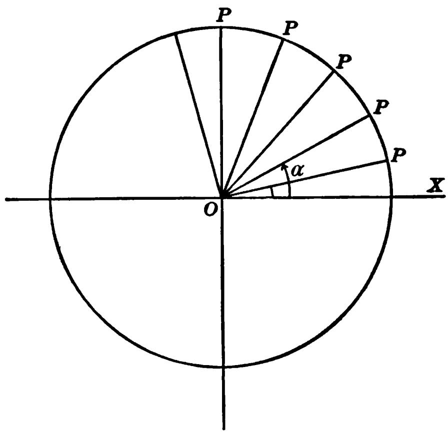


Fig. 110

As the angle α increases, the point P will move around the circumference of the circle, from a position on the X-axis, when

the angle $\alpha = 0°$, through the four quadrants, till it reaches the same position again, when $\alpha = 360°$. The question is, What happens to $\sin \alpha$ as α passes through this series of values? When $\alpha = 0°$, the ordinate of P is 0, and hence the ratio $\dfrac{\text{ordinate}}{\text{radius vector}} = 0$; that is, $\sin 0° = 0$. Now as α increases, the ordinate increases also; and since the radius vector is constant, $\sin \alpha$ must increase. This increase of the ordinate, and therefore of the sine function, continues through the first quadrant, until, when the angle is 90°, the ordinate equals the radius vector, and therefore $\sin 90° = 1$. As the angle increases from 90° to 180°, the sine of the angle evidently decreases from 1 to 0; as the angle passes through the third quadrant, $\sin \alpha$ decreases from 0 to -1; and, finally, as α increases from 270° to 360°, $\sin \alpha$ increases from -1 to 0.

138. This review of the variation in the sine function should be completed by drawing up a table of the values of the function for all angles at 15° intervals, from 0° to 360°. Do not use the printed tables for this purpose, but take the results obtained from our previous study (cf. pp. 68, 164). When this table of values has been written down, it will be found easy to construct the graph of the function $y = \sin x$ from $x = 0$ to $x = 360$ by taking the values of the angle (x) as abscissas and the corresponding values of $\sin x$ as ordinates. Plotting the points located by the table in this way, and joining them by a smooth curve, we shall have a part of the graph of the function. Fig. 111 gives a small-scale representation of the curve obtained. The student should make a careful drawing on a larger scale. (The scale on the Y-axis will need to be much larger than that on the X-axis, for obvious practical reasons.)

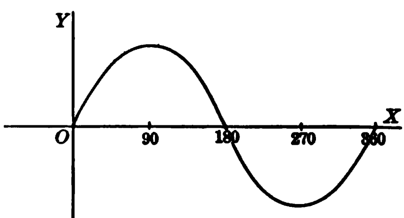


FIG. 111

139. Since the increase of an angle beyond 360° means merely increasing the amount of rotation beyond one complete revolution, the sine function will change in precisely the same way while the

angle changes from 360° to 450° as it does while the angle changes from 0° to 90°; in other words, beyond 360° the same cycle of changes is repeated, because the terminal line of the angle merely occupies the same positions again, in the same order. Thus the sine function repeats itself periodically at intervals of 360°. It is accordingly called a *periodic* function, with the *period* 360°. Anticipating the next paragraphs, we can see that all the trigonometric functions are periodic, with a period 360°. For *negative* values of the angle we have the fact that $\sin(-x) = -\sin x$, which enables us to extend the drawing of the sine graph to the left of the Y-axis. This should be done in the final construction of the curve.

140. The cosine. By similar reasoning to that for the sine, and by referring to Fig. 110 again, the student may show that the cosine varies as follows:

Angle	0° to 90°	90° to 180°	180° to 270°	270° to 360°
Cosine	1 to 0	0 to −1	−1 to 0	0 to 1

And, as in the case of the sine, a table of values of the cosine function should be drawn up for angles at 15° intervals, from 0° to 360°. From this table points can be plotted and the graph of $y = \cos x$ drawn.

141. The tangent. Since the tangent of an angle equals the ratio $\dfrac{\text{ordinate}}{\text{abscissa}}$, we see at once that $\tan 0° = 0$ and that, as the angle increases, the tangent also increases through the first quadrant. For if we keep the *abscissa* constant, as in Fig. 112, the ordinate will evidently increase as the angle increases. As the angle α approaches 90°, the ordinate of the point Q, where the terminal line of the angle meets the line AP_1,

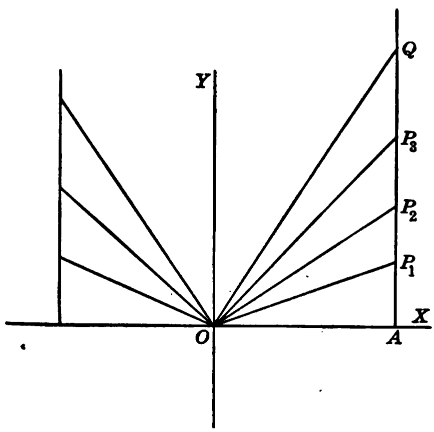


FIG. 112

grows larger and larger. This ordinate will eventually exceed any value that can be named if the angle α be taken sufficiently near to 90°; and thus $\tan \alpha$ will exceed any assignable value if α is near enough to 90°. If $\alpha = 90°$, however, $\tan \alpha$ does not exist, since the ratio $\frac{\text{ordinate}}{\text{abscissa}}$ would then take the form $\frac{\text{ordinate}}{0}$ (the abscissa of any point on the Y-axis being 0), and a fraction whose denominator equals zero has no value whatever.[1] The fact that the tangent function will exceed any assignable value, for angles in the vicinity of 90°, is expressed by saying, "$\tan \alpha$ becomes infinite as α approaches 90°," or, in symbols, $\tan 90° = \infty$. (This may be read "equals infinity," but it is not to be understood as giving a *value* to $\tan 90°$, which, as we have just seen, is impossible; on the contrary, it only expresses in brief symbolic form the fact that the tangent of an angle will exceed any assignable value if the angle is near enough to 90°.)

As soon as the angle α passes into the second quadrant, the tangent is *negative* and numerically very large. Thus there is an enormous jump in the value of the function as the angle passes from a value a little less than 90° to one a little more than 90°. Contrast this behavior of the tangent function with that of the sine, which changes *gradually* or *continuously* as the angle changes from a value a little less than 90° to one a little greater than 90°. This is an example of the important distinction between a function which is *continuous* at a point (as the sine at 90°) and one which is *discontinuous* at a point (as the tangent at 90°). Of course it is obvious that both functions are continuous for any value of the angle *between* 0° and 90°.

Returning to the variation of the tangent: as the angle α increases from 90° through the second quadrant, Fig. 112 shows that the *length* of the ordinate decreases (the abscissa being kept constant), and hence $\tan \alpha$ decreases *numerically*; but as its value is negative, it actually *increases* from $-\infty$ to 0. (The symbol $-\infty$

[1] Notice that it is a very different thing to say that a certain expression "has no value" and to say it "equals zero." Zero *is* a perfectly definite value (cf. note on page 8).

is to be interpreted in accordance with what was said above concerning the symbol ∞.) As the angle increases from 180° to 270°, the tangent becomes positive and increases from 0 to ∞. Is it continuous or discontinuous at 180? at 270? Finally, as the angle increases from 270° to 360°, the tangent increases from $-\infty$ to 0.

142. Remembering that $\tan(180° + \theta) = \tan\theta$, we see that this function has a period of 180°, unlike the functions sine and cosine, which do not repeat their cycle of values for a smaller interval than 360°. The student should now make a table of values of $\tan x$ for all values of x at intervals of 15°, from 0° to 360°, and make the graph of the function $y = \tan x$. Fig. 113 gives a small-scale representation of a part of this curve, which should be drawn on a large scale, with considerable accuracy.

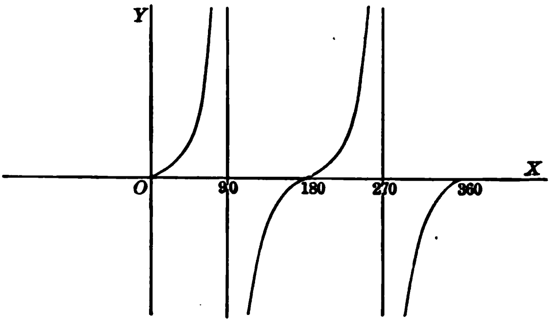


Fig. 113

143. The graphical representation of these functions, as in the case of all others that we have studied, is a great help in forming definite and accurate ideas of the way in which the functional values change. To gain more detailed knowledge of these changes one must study that branch of mathematics which is known as Differential Calculus (see Chapter XI). That study enables us to answer questions about the *rate* of increase or decrease of functional values, whereas for the present we must be satisfied with the general information which the graph gives.

EXERCISE

Study in the same way the variations in the other three trigonometric functions, and draw their graphs.

POLAR COÖRDINATES

***144.** We have seen how any point can be located by means of its distances from two perpendicular straight lines; it is also possible to locate a point in various other ways, which are found useful in solving a great many problems. The most important of these other ways is by means of the *radius vector* of the point and the *angle which the radius vector makes with a fixed line.* Thus, the point P is located by the radius vector r and the angle θ, which is called the *vectorial angle* of P. The line OA is called the *initial line,* and the angle θ may have any value, positive or negative. The values (r, θ) are called the *polar coördinates* of the point P.

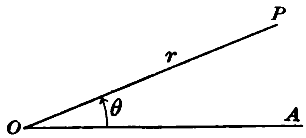


FIG. 114

Thus, in Fig. 115 the point P is completely located by the radius vector 2 and the vectorial angle 25°, which values are its polar coördinates. The radius vector is always mentioned first, so that the statement $P \equiv (2, 25°)$ locates definitely the point P.

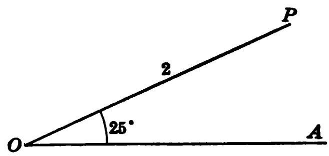


FIG. 115

Similarly, $Q \equiv (\frac{1}{2}, 200°)$ locates the point Q in Fig. 116, and $Q \equiv (\frac{1}{2}, -160°)$ locates the same point, as does also $Q \equiv (\frac{1}{2}, 560°)$. It is thus clear that a point can have an indefinite number of pairs of polar coördinates (but a single pair of coördinates determines only *one* point). Moreover, it is customary to consider that a *negative* radius vector gives a point at the same distance from the origin as the corresponding positive radius vector, but measured *in exactly the opposite direction.* Thus, the point P in Fig. 115 is not only (2, 25°) but also (− 2, 205°), and Q in Fig. 116 is $(-\frac{1}{2}, 20°)$ or $(-\frac{1}{2}, 380°)$.

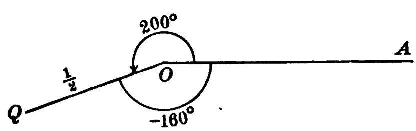


FIG. 116

EXERCISES

1. Locate each of the following points: $(1, 45°)$; $(5, 90°)$; $(2, 0°)$; $(\frac{1}{3}, 270°)$; $(-1, 100°)$; $(2, 320°)$. Choose also other pairs of coördinates at random and locate the corresponding points.

2. Where can a point be if its radius vector is 2? if its vectorial angle is 180°? 0°?

3. Prove that the line joining the points $(1, 45°)$ and $(1, 135°)$ is parallel to the initial line.

4. Show that the points $(0, 0)$, $(2, 30°)$, and $(2, 90°)$ form an equilateral triangle, and find the (polar) coördinates of the mid-points of its sides.

5. The rectangular coördinates of a point (that is, its abscissa and ordinate) are $(2\sqrt{2}, 2\sqrt{2})$. Find its polar coördinates. Answer the same question for the point $(-2\sqrt{2}, 2\sqrt{2})$; for the point $(2\sqrt{2}, -2\sqrt{2})$.

***145.** If (r, θ) represent variable coördinates, r being a function of θ expressed by the equation $r = f(\theta)$, then the corresponding point will take various positions, the totality of which form the *graph of the function*

$$r = f(\theta).$$

The following examples will illustrate the way in which such graphs are studied.

Example 1. $r = \frac{1}{4}\theta$. Making a table of corresponding values of r and θ, we have

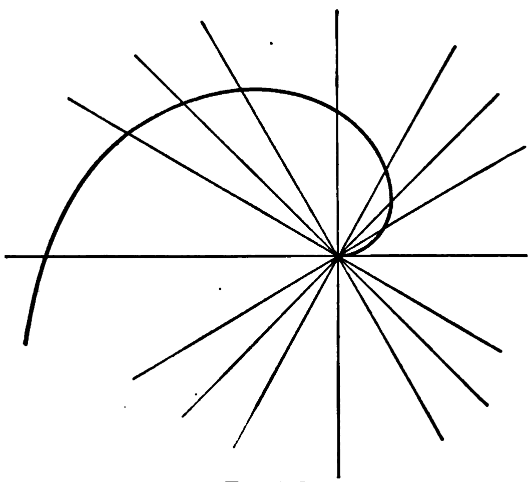


FIG. 117

θ	0	30°	45°	60°	75°	90°	135°	180°	etc.
r	0	$7\frac{1}{2}$	$11\frac{1}{4}$	15	$18\frac{3}{4}$	$22\frac{1}{2}$	$33\frac{3}{4}$	45	

Evidently the graph is a spiral, as illustrated in Fig. 117. It is known as the Spiral of Archimedes.

Example 2. $r = \sin \theta$.

Making a table of values of r and θ,

r	0	$\frac{1}{2}$	.87	1	.87	$\frac{1}{2}$	0	$-\frac{1}{2}$	
θ	0	30°	60°	90°	120°	150°	180°	210°	etc.

Evidently all values of θ in the third or fourth quadrant will give negative values of r, and hence the corresponding points will be in the first or second

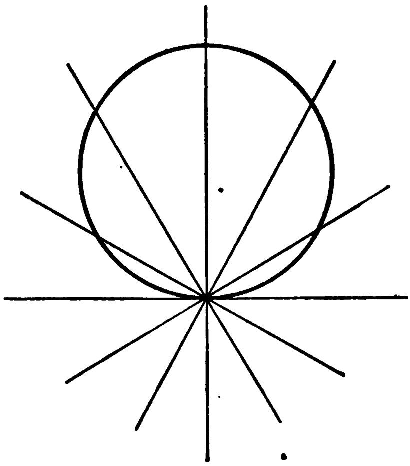


FIG. 118

quadrant. Furthermore, $\sin(90° + \theta) = \sin(90° - \theta)$, so that the graph is symmetrical with respect to the line $\theta = 90°$. The curve is a circle (Fig. 118).

Example 3. $r = \sin 2\theta$.

Here the table of values is as follows:

r	0	.5	.87	1	.87	.5	0	$-.5$
θ	0	15°	30°	45°	60°	75°	90°	105°

r	$-.87$	-1	$-.87$	$-.5$	0	.5	.87	
θ	120°	135°	150°	165°	180°	195°	210°	etc.

in the same order as for θ in the first and second quadrants.

We do not need to pay attention to the values of θ beyond 180°, because increasing θ by 180° increases 2θ by 360°, and hence gives the same value

of $\sin 2\theta$ as if we had used θ itself. Thus, $\sin(2 \cdot 195°) = \sin(2 \cdot 15°)$, $\sin(2 \cdot 210°) = \sin 60°$, etc. Of course the values of r are negative for all values of θ in the second quadrant (for 2θ is then in the third or fourth

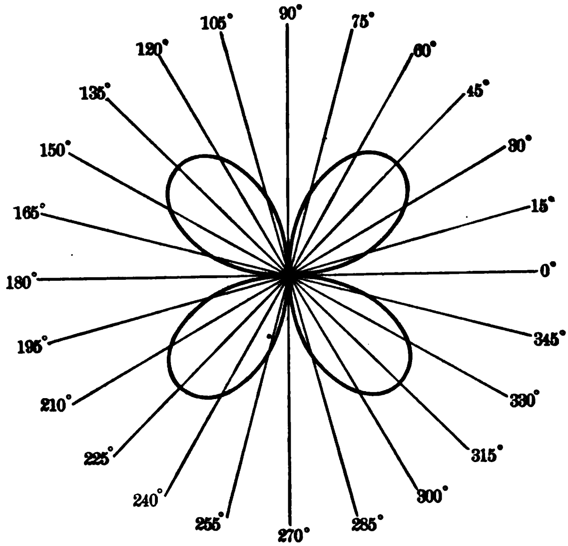


FIG. 119

quadrant); hence the points on the curve are in the *fourth* quadrant (see Fig. 119). When θ is in the third quadrant, however, 2θ is in the first or second; hence r is *positive*, and the points on the curve are in the third quadrant. The graph is the curve known as the four-leaved rose.

EXERCISES

Draw the graphs of each of the following equations in polar coördinates:

1. $r = \theta$.
2. $r = \cos\theta$.
3. $r = 1 - \cos\theta$.
4. $r = \cos 2\theta$.
5. $r = \tan\theta$.
6. $r^2 = \cos 2\theta$.
7. $r = \sin\dfrac{\theta}{2}$.
8. $r = \tan\dfrac{\theta}{3}$.
9. $r = \tan\theta \cdot \sin\theta$.
10. $r = \dfrac{1}{\theta}$.
11. $r = \sin\dfrac{\theta}{2} + 1$.
12. $r = \sec\theta \pm 1$.
13. $r = 1 + \sin\frac{3}{2}\theta$.
14. $r = \dfrac{6}{1 - 2\cos\theta}$.

15. $r = \dfrac{8}{1 + 2\cos\theta}$.

16. $r = a\cos^3\dfrac{\theta}{3}$.

17. $r\cos 2\theta = a^2$.

18. $r = \dfrac{3}{1 + \cos\theta}$.

19. $r = \dfrac{8}{3 - \cos\theta}$.

20. $r = 4(1 - 2\cos\theta)$.

21. $r = a\sin^3\dfrac{\theta}{3}$.

22. $r = a\csc^2\dfrac{\theta}{2}$.

23. $r = a\sin^2\dfrac{\theta}{2}$.

24. $r = a\tan^2\theta\sec\theta$.

25. $r = a\sin 3\theta$.

26. $r = a\cos 3\theta$.

27. $r^2 = \cos 4\theta$.

28. $r^2\cos\theta = a^3\sin 3\theta$.

29. $r = \sin\dfrac{\theta}{2} + \cos\dfrac{\theta}{2}$.

30. $r^2 = \dfrac{a^2b^2}{a^2\sin^2\theta + b^2\cos^2\theta}$.

31. $r = a\tan 3\theta$.

$\cos^2 a - \cot^2 a =$

CHAPTER X

THE EXPONENTIAL AND LOGARITHMIC FUNCTIONS

146. In elementary algebra the symbol a^n was defined, where n represents a positive integer, as meaning $a \cdot a \cdot a \cdot a \cdots$ to n factors. The number a is called the *base*, and n the *exponent.* The laws of operation are as follows:

$$\text{I.}\quad a^h \cdot a^k = a^{h+k}.$$

$$\text{II.}\quad \frac{a^h}{a^k} = a^{h-k} \text{ if } h > k$$
$$= \frac{1}{a^{k-h}} \text{ if } k > h.$$

$$\text{III.}\quad (a^h)^k = a^{kh}.$$

$$\text{IV.}\quad (ab)^h = a^h \cdot b^h.$$

EXERCISES

The following easy exercises in review of these important laws should be worked over as often as is necessary in order to insure perfect mastery of them:

1. $\dfrac{x^3y^2}{x^2y^3} = ?\quad \dfrac{(a+b)^5}{a+b} = ?\quad \dfrac{(x^2-y^2)^2}{(x-y)^2} = ?$

2. $2^{10} \cdot 2 = ?\quad \dfrac{2^{10}}{2} = ?\quad 2^{10} + 2 = ?$ (Do *not* multiply out.)

3. $\dfrac{(x^2+y^2)^8}{(x^4-y^4)^4} = ?\quad \dfrac{[(a+b)^2-1]^3}{(a+b-1)^3} = ?$

4. $x^n \cdot x^{2n} = ?\quad a^x \cdot a^y = ?$

5. $\dfrac{a^{2n-1}}{a} = ?\quad \dfrac{a^{2n-1}}{a^n} = ?\quad \dfrac{a^{2n-1}}{a^{2n}} = ?\quad \dfrac{a^{2n-1}}{a^{2n+1}} = ?$

6. $\dfrac{(x^2-y^2)^n}{(x+y)^n} = ?\quad [(a+b)^n]^n = ?\quad [(a+b)^3]^2 = ?$

7. What is the value of $(a^3)^2$? of a^{3^2}? (Note that $(a^m)^n$ is *not* the same as a^{m^n}.)

8. What is the difference in meaning between 2^{2^3} and $(2^2)^3$?

147. It is evident that our definition of a^n has no meaning unless n is a positive integer, because we cannot multiply together a *fractional* number of factors or a *negative* number of them; but it is equally evident, after a little thought, that the *laws* I–IV do *not* lead to any absurdity if we suppose that the exponent can become fractional or negative. For instance, Law I gives for $h = k = \frac{1}{2}$, $a^{\frac{1}{2}} \cdot a^{\frac{1}{2}} = a^1$; that is, $a^{\frac{1}{2}}$ is a *number which, when multiplied by itself, gives a.*

But "a number which, when multiplied by itself, gives a" is the *square root of a.* Hence Law I will be absolutely true for $h = k = \frac{1}{2}$ if we simply regard $a^{\frac{1}{2}}$ as meaning $\sqrt{a}$. We shall accordingly take this as the definition of $a^{\frac{1}{2}}$. (Notice that Law III leads to the same result, by taking $h = \frac{1}{2}$, $k = 2$.) By using the same process of reasoning with three factors we have $a^{\frac{1}{3}} \cdot a^{\frac{1}{3}} \cdot a^{\frac{1}{3}} = a$, and hence we must define $a^{\frac{1}{3}}$ as meaning $\sqrt[3]{a}$; and by extending the method farther we find, in general, $a^{\frac{1}{n}} = \sqrt[n]{a}$. (Or, still better, we may get this result by using Law III with $h = \dfrac{1}{n}$ and $k = n$.)

EXERCISES

1. Show (from Law III) that $a^{\frac{2}{3}} = (\sqrt[3]{a})^2 = \sqrt[3]{a^2}$; that $a^{\frac{3}{2}} = \sqrt{a^3}$; that $a^{\frac{3}{4}} = \sqrt[4]{a^3}$; and, in general, that $a^{\frac{m}{n}} = \sqrt[n]{a^m} = (\sqrt[n]{a})^m$.

2. Write down the values of $9^{\frac{3}{2}}$, $8^{\frac{2}{3}}$, $4^{\frac{5}{2}}$, $16^{\frac{3}{4}}$.

148. Now let us proceed in the same way to find a meaning for a negative exponent by testing the laws I–IV. For instance, what should be the meaning of a^{-1}? If Law I is to hold, $a^{-1} \cdot a^2 = a^{2-1} = a$.

Therefore $$a^{-1} = \frac{a}{a^2} = \frac{1}{a}.$$

Or h can be kept *general*:

$$a^h \cdot a^{-1} = a^{h-1}.$$

Therefore $$a^{-1} = \frac{a^{h-1}}{a^h} = \frac{1}{a}.$$

Similarly, it can be shown that

$$a^{-n} = \frac{1}{a^n}.$$

EXERCISES

1. Show that the last statement is true whether n is an integer or a fraction.

2. Write down the values of 4^{-2}, $4^{-\frac{1}{2}}$, $4^{-\frac{3}{2}}$, 10^{-1}, 10^{-3}, $9^{-\frac{5}{2}}$, $8^{-\frac{1}{3}}$, $8^{-\frac{2}{3}}$, $8^{-\frac{4}{3}}$, $16^{-\frac{3}{4}}$.

3. Show that the first part of Law II holds even if $h < k$. What does it become if $h = k$? What is therefore the meaning that we must give to a^0?

4. Make a table of the powers of the base 4, using as exponents all values that give *rational* results, from 4^{-6} to 4^6, inclusive. The table should then have twenty-five entries. By its aid many problems of multiplication and division can be solved without any computation.

Thus, $$32 \times 16 = ?$$

Solution. From the table,
$$32 = 4^{\frac{5}{2}},$$
$$16 = 4^2.$$

Therefore $$32 \times 16 = 4^{\frac{5}{2}} \cdot 4^2 = 4^{\frac{9}{2}} = 512.$$

Again, $$4096 \div 128 = ?$$

Solution. From the table,
$$4096 = 4^6,$$
$$128 = 4^{\frac{7}{2}}.$$

Therefore $$4096 \div 128 = 4^{6-\frac{7}{2}} = 4^{\frac{5}{2}} = 32.$$

Work the following in a similar way:

(1) $8 \div 128 = ?$ (5) $\sqrt[3]{4048} = ?$ (9) $\sqrt{\dfrac{32.64}{2.256}} = ?$

(2) $256 \cdot 8 = ?$ (6) $\sqrt{1024} = ?$

(3) $\frac{1}{32} \cdot 2048 = ?$ (7) $\sqrt[5]{1024} = ?$ (10) $(\frac{1}{16} \cdot 128)^{\frac{5}{3}} = ?$

(4) $\frac{1}{512} \cdot 64 = ?$ (8) $\sqrt{256} \cdot \sqrt[3]{512} = ?$

149. Such a table as that of Ex. 4 is called a table of *logarithms.* For instance, in the equation $4^3 = 64$, 3 is called the *logarithm* of 64 to the base 4. In general, *if* $a^x = b$, *x is the logarithm of b to the base a.* This is abbreviated $x = \log_a b$. The number b in this equation is called the *antilogarithm.* Thus a *logarithm* is the exponent of the power to which the *base* must be raised in order to get the *antilogarithm.* "$8^{-\frac{1}{3}} = \frac{1}{2}$" means

exactly the same as "$\log_8 \frac{1}{2} = -\frac{1}{3}$." 8 is the base, $-\frac{1}{3}$ the logarithm, and $\frac{1}{2}$ the antilogarithm. The student should now practice translating equations from the exponential form to the logarithmic and back again.

EXERCISES

$$8^{\frac{1}{3}} = 2 \text{ is equivalent to } \log_8 2 = \tfrac{1}{3}.$$

$$\log_{10} 100 = 2 \text{ is equivalent to } 10^2 = 100.$$

1. Work the following in a similar way:

(1) $\log_3 9 = ?$	(6) $\log_9 3 = ?$	(11) $\log_{16} 8 = ?$
(2) $\log_{10} \frac{1}{100} = ?$	(7) $\log_{27} (\frac{1}{9}) = ?$	(12) $\log_{27} 81 = ?$
(3) $\log_{100} 10 = ?$	(8) $\log_{10} 100 = ?$	(13) $\log_{100} 10 \cdot \log_{10} 100 = ?$
(4) $\log_3 27 = ?$	(9) $\log_{27} 3 = ?$	(14) $\log_3 27 \cdot \log_{27} 3 = ?$
(5) $\log_8 16 \cdot \log_{16} 8 = ?$	(10) $\log_5 125 \cdot \log_{125} 5 = ?$	

2. Translate into the logarithmic form,

(1) $10^3 = 1000.$ (2) $16^{-\frac{3}{4}} = \frac{1}{8}.$ (3) $32^{-\frac{4}{5}} = \frac{1}{16}.$

3. What is the value of $\log_{10} 1$? of $\log_a 1$?

150. Having thus fixed in mind the fundamental fact that *logarithms are exponents,* we see that the laws of operation with exponents must naturally give corresponding laws of operation with logarithms. Law I, $a^h \cdot a^k = a^{h+k}$, tells us that the exponent of a product is equal to the sum of the exponents of the factors, the base being the same. Hence, using the word "logarithm" for its equivalent, "exponent," we have, as the First Law of Logarithms, *The logarithm of a product equals the sum of the logarithms of the factors, all to the same base.*

151. Similarly, Law II, $\dfrac{a^h}{a^k} = a^{h-k}$, tells us that, the base being the same, the exponent in a quotient equals the exponent in the dividend minus the exponent in the divisor. Hence, using the word "logarithm" for "exponent," we have, as the Second Law of Logarithms, *The logarithm of a quotient equals the logarithm of the dividend minus the logarithm of the divisor.*

152. Similarly, Law III, $(a^h)^k = a^{h \cdot k}$, gives as the Third Law of Logarithms, *The logarithm of a power of a number equals the logarithm of the number, multiplied by the exponent of the power.* For, calling

$$a^h = N, \tag{1}$$

$$N^k = a^{hk}. \quad \text{(Law III)} \tag{2}$$

Translating (1) into logarithmic form,

$$h = \log_a N.$$

Translating (2) into logarithmic form,

$$hk = \log_a (N^k).$$

Therefore $$\log_a (N^k) = k \cdot \log_a N.$$

Example. $\log_8 (4^3) = 3 \log_8 4$.

We can verify this by writing down the values of the logarithms on both sides of this equation, thus:

$$4^3 = 64,$$

$$\log_8 64 = 2;$$

and $$\log_8 4 = \tfrac{2}{3},$$

$$3 \log_8 4 = 3 \cdot \tfrac{2}{3} = 2 = \log_8 64,$$

as was to be shown.

153. Since the nth root is the same as the $\frac{1}{n}$th power, this law also includes as a corollary the following: *The logarithm of a root of a number equals the logarithm of the number, divided by the index of the root.*

154. If we now consider the application of these laws of logarithms (taking for base some number > 1), we see at once that the tables we can make are very restricted in their scope. Thus, with the base 4, or the base 8, we can easily give the values of $\log 2$, $\log 4$, $\log \frac{1}{2}$, etc., but not of $\log 3$ or of $\log 5$; and with the base 9 we can give the value of $\log 3$, but not that of $\log 2$. We do not even know that there *is* any power of 4 that will give 3, or of 9 that will give 2. Thus, no matter what base we take, there will be large gaps in our table.

How can this incompleteness be overcome? Only by introducing an *assumption*, which, although it cannot now be proved,

yet leads to consistent results and is of the very highest utility. This assumption is, *With any base greater than 0, except 1, every positive number has a logarithm*; that is, there exists a number x such that $a^x = N$, where N and a are any positive numbers ($a \neq 1$). It turns out that this logarithm x is in most cases an *irrational number*; thus, $\log_4 3$ and $\log_9 2$ are irrational. To find their *approximate* value as rational numbers is a problem of no very great difficulty; in fact, it is not much harder than extracting the cube root of numbers by the arithmetical method. This computation of logarithms, however, is best taken up at a later point in the mathematical course; for the present we shall use the results obtained by other computers and published in tables of logarithms. (For further information see the articles "Tables" and "Logarithms" in the Encyclopedia Britannica.)

For practical purposes the base 10 is the most convenient. We shall therefore use that base altogether, so that, unless otherwise stated, $\log N$ is hereafter to be understood as meaning $\log_{10} N$.

EXERCISE

Make a table of the positive and negative integral powers of 10, from 10^6 to 10^{-6}, and translate each item into its equivalent logarithmic equation, thus: $10^1 = 10$, $\log 10 = 1$, etc.

155. Using this elementary table, it will be seen that the logarithm of any (positive) number can be determined to the *nearest integer* at a glance. Thus, log 11, log 12, and the logarithm of *any* number between 10 and 100 will be between 1 and 2, that is, will equal 1 plus a decimal; the logarithm of any number between 100 and 1000 will be between 2 and 3, that is, will equal 2 plus a decimal; and so on. The decimal part of the logarithm is called the *mantissa*; the integral part, the *characteristic*. Using these words, we have at once the First Rule: *The characteristic of the logarithm of a number between 1 and 10 is 0, of a number between 10 and 100 is 1, of a number between 100 and 1000 is 2, and, in general, is* ***one less than the number of digits to the left of the decimal point.*** But if there are *no* significant figures to the left of the decimal

point (that is, if the antilogarithm is less than 1), another rule must be formulated. Noting that the characteristic of the logarithm of a number between .1 and 1 is -1 (since the logarithm is then -1 plus a decimal), that the characteristic of the logarithm of a number between .01 and .1 is -2, and so on, we have the Second Rule: *When a number is less than 1, the characteristic of its logarithm is* ***negative*** *and numerically* ***one more than the number of zeros after the decimal point before the first significant figure.***

EXERCISE

Determine the characteristic of the logarithm of each of the following numbers: 531, 24, .62, 7.1, .006, 4.06, .05001, 56.3, .403, .9, 45,000,000.

156. For the mantissa we use printed tables of logarithms, as has been said; but their use is facilitated by the following principle: *The mantissa of a logarithm is not changed by moving the decimal point in the corresponding number*; that is, log 2, log .2, log .0002, log 200, log 20,000,000, all have the *same mantissa.* This important fact is very easily proved, thus: Given $\log N$, then we have, from §§ 150 and 151,

$$\log(10\,N) = \log N + \log 10 = (\log N) + 1;$$

and
$$\log\left(\frac{N}{10}\right) = \log N - \log 10 = (\log N) - 1;$$

in general, $\log(N \cdot 10^k) = (\log N) + k,$

k being an integer (positive or negative).

Example. Log 2 = 0.3010, log 20 = 1.3010, log 2000 = 3.3010, log .2 = -1 + .3010 (written $\bar{1}.3010$), log .00002 = -5 + .3010 = $\bar{5}.3010$.

157. We have now the ability to find, with the help of the table, the (approximate) values of the logarithms of all positive numbers. The student should practice doing this until the use of the printed tables becomes very easy. Then it is time to apply the laws of logarithms to problems. A few examples will first be worked out.

Example 1. Find the product of 36 by 124.

Solution. By the First Law, $\log(36 \cdot 124) = \log 36 + \log 124.$

$$\begin{array}{llll}
\text{From the table,} & \log 36 = 1.5563 & \text{that is,} & 10^{1.5563} = 36 \\
\text{Also} & \log 124 = \underline{2.0934} & \text{that is,} & \underline{10^{2.0934} = 124} \\
 & \log(36 \cdot 124) = 3.6497 & & 10^{1.5563+2.0934} = 36 \cdot 124 \\
\text{Therefore} & 36 \cdot 124 = \underline{\underline{4464}} & & \text{By the table, } 10^{3.6497} = 4464
\end{array}$$

Example 2. Find the quotient of .0031 by .0925.

Solution. Write
$$x = \frac{.0031}{.0925};$$

then
$$\log x = \log .0031 - \log .0925. \quad \text{(Second Law)}$$

$$\begin{array}{lrl}
\text{From the table,} & \log .0031 = & \bar{3}.4914 \\
 & \log .0925 = & \underline{\bar{2}.9661} \\
 & \log x = & \bar{2}.5253 \qquad \text{Therefore } x = \underline{\underline{.03352}}
\end{array}$$

Translate each equation also into the exponential form, as was done in Example 1.

Example 3. Find the value of $\sqrt{3}$ to three decimal places.

Solution. Let
$$x = \sqrt{3};$$

then
$$\log x = \tfrac{1}{2} \log 3,$$

by the Corollary to Law III (§ 153).

From the table, $\log 3 = 0.4771.$

Therefore $\log x = 0.2385,$

and $x = \underline{\underline{1.732}},$

correct to four figures, that is, to three decimal places.

Example 4. Find the value of $\dfrac{213 \times 3.23}{\sqrt[3]{594}}$.

Solution. If x is the required value, $\log x = \log(\text{dividend}) - \log(\text{divisor})$ $= \log 213 + \log 3.23 - \frac{1}{3} \log 594.$

$$\begin{array}{lrl}
\text{We find} & \log 213 = & 2.3284 \\
 & \log 3.23 = & \underline{0.5092} \\
 & & 2.8376 \\
\text{Log } 594 = 2.7738, & \tfrac{1}{3}\log 594 = & \underline{0.9246} \\
 & \log x = & 1.9130 \qquad \text{Therefore } x = \underline{\underline{81.84}}+
\end{array}$$

EXERCISES

Find (by using logarithms) the value of each of the following expressions:

1. $3.14 \times .216$.
2. $\dfrac{293}{.4772}$.
3. $\sqrt{21.5}$.
4. $\sqrt[3]{1.331}$.
5. $(1.01)^{15}$.
6. $\dfrac{9.003 \times .3874}{25 \times .0067}$.
7. $\sqrt{14 \times 21 \times 35 \times 112 \times 3.42}$.

8. Find the area of a circle of radius $2\frac{1}{4}$ in. (Take $\log \pi = 0.4971$.)

9. Find the volume of a sphere of radius 3 ft. 3 in.; find also its weight if its specific gravity is 7.2 and a cubic foot of water weighs 62.5 lb.

10. Find the amount of \$1 at compound interest for 10 yr. at 6%; for 20 yr.; for 100 yr.

HINT. The amount at the end of 1 yr. is \$1.06; at the end of 2 yr., $\$1.06 \cdot 1.06 = \$1.06^2$; and so on.

11. Use logarithms to find the unknown parts and the area of the $\triangle ABC$, given $a = 15.63$, $\alpha = 26° 10'$, $\gamma = 90°$.

Solution. $$\frac{b}{a} = \cot \alpha.$$

Therefore $$b = a \cot \alpha.$$

Therefore $$\log b = \log a + \log \cot \alpha.$$

Also $$c = \frac{a}{\sin \alpha}.$$

Therefore $$\log c = \log a - \log \sin \alpha.$$

$$\begin{aligned} \log a &= 1.1940 \\ \log \cot \alpha &= \underline{0.3086} \\ \log b &= 1.5026 \\ b &= \underline{\underline{31.81}} \end{aligned} \qquad \begin{aligned} \log a &= 1.1940 \\ \log \sin \alpha &= \underline{9.6444 - 10} \\ \log c &= 1.5496 \\ c &= \underline{\underline{35.45}} \end{aligned}$$

Check. $c^2 - b^2 = a^2 = (c + b)(c - b)$.

$$c + b = 67.26.$$
$$c - b = \ \ 3.64.$$

$$\begin{aligned} \log (c + b) &= 1.8278 \\ \log (c - b) &= \underline{0.5611} \\ \log a^2 &= 2.3889 \end{aligned}$$

$\log a = 1.1944$, which checks the above results.

For the area, $S = \dfrac{ab}{2}.$

$$\begin{aligned}\log a &= 1.1940\\ \log b &= 1.5026\\ \log \tfrac{1}{2} &= \underline{9.6990-10}\\ \log S &= 2.3956\end{aligned}$$

Therefore the area is $\underline{\underline{248.6^+}}$.

NOTE. The student will observe that the logarithms of the trigonometric functions are given in a separate table. Negative characteristics are usually written "$9-10$" for "-1," and so on.

12. Use logarithms to solve, check, and find the area, given

(1) $a = 421,\quad \beta = 54° 35',\quad \gamma = 90°.$
(2) $\alpha = 37° 15',\quad \beta = 71° 54',\quad a = 4.263.$
(3) $\beta = 115°,\quad \gamma = 30° 30',\quad a = 1.001.$
(4) $\alpha = 10° 40',\quad \gamma = 150° 10',\quad c = .124.$
(5) $\alpha = 41° 34',\quad a = 46.70,\quad b = 60.03.$
(6) $a = .698,\quad c = .615,\quad \gamma = 38° 15'.$
(7) $a = 3.16,\quad b = 5.09,\quad \beta = 147° 59'.$

158. These last problems make it clear that the use of logarithms is practical in solving triangles by the use of the Law of Sines; but it is evidently impossible to use them to so good advantage in cases where the Law of Cosines applies, because the Law of Cosines contains *additions* as well as multiplications, and these cannot be performed by logarithms. So it is necessary, in order to gain the advantage of the use of logarithms in the computations, to obtain modified formulas that shall include *no additions or subtractions*. We begin with *three sides given.*

By the Law of Cosines,

$$\cos \alpha = \frac{b^2 + c^2 - a^2}{2\,bc};$$

but we have also $\cos \alpha = 1 - 2\sin^2\dfrac{\alpha}{2} = 2\cos^2\dfrac{\alpha}{2} - 1.$ (Ex. 3, p. 119)

Therefore
$$\begin{aligned}2\sin^2\frac{\alpha}{2} &= 1 - \cos\alpha = 1 - \frac{b^2 + c^2 - a^2}{2\,bc}\\ &= \frac{2\,bc - b^2 - c^2 + a^2}{2\,bc} = \frac{a^2 - (b-c)^2}{2\,bc}\\ &= \frac{(a+b-c)(a-b+c)}{2\,bc}.\end{aligned}$$

The numbers $a+b-c$ and $a-b+c$ are very easily found, and so we have a practical formula for logarithmic computation:

$$\sin\frac{1}{2}\alpha=\sqrt{\frac{(a+b-c)(a-b+c)}{4\,bc}}. \tag{1}$$

It is usual to abbreviate by writing $2\,s=a+b+c$, or

$$s=\frac{a+b+c}{2}.$$

Therefore
$$2\,s-2\,a=b+c-a,$$
$$2\,s-2\,b=a-b+c,$$
$$2\,s-2\,c=a+b-c.$$

Therefore (1) becomes $\sin\frac{1}{2}\alpha=\sqrt{\frac{(s-b)(s-c)}{bc}}.$ (1′)

Similarly, by starting with $2\cos^2\frac{1}{2}\alpha=1+\cos\alpha$, let the student show that

$$\cos\frac{1}{2}\alpha=\sqrt{\frac{s(s-a)}{bc}}. \tag{2}$$

Dividing (1′) by (2), $\tan\frac{1}{2}\alpha=\sqrt{\frac{(s-b)(s-c)}{s(s-a)}}.$ (3)

Using the other angles of the triangle instead of α,

$$\tan\frac{1}{2}\beta=\sqrt{\frac{(s-a)(s-c)}{s(s-b)}}.$$

$$\tan\frac{1}{2}\gamma=\sqrt{\frac{(s-a)(s-b)}{s(s-c)}}.$$

The formulas for the tangent are generally the best to use, because all three angles can be found with very little more work than would otherwise be needed to find one angle.

One further abbreviation is very useful:

Since
$$\tan\frac{1}{2}\alpha=\frac{1}{s-a}\sqrt{\frac{(s-a)(s-b)(s-c)}{s}},$$

and
$$\tan\frac{1}{2}\beta=\frac{1}{s-b}\sqrt{\frac{(s-a)(s-b)(s-c)}{s}},$$

and
$$\tan\frac{1}{2}\gamma=\frac{1}{s-c}\sqrt{\frac{(s-a)(s-b)(s-c)}{s}},$$

therefore, if we write

$$r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}},$$

we have

$$\tan\frac{1}{2}\alpha = \frac{r}{s-a},$$

$$\tan\frac{1}{2}\beta = \frac{r}{s-b},$$

$$\tan\frac{1}{2}\gamma = \frac{r}{s-c};$$

or, using logarithms,

$$\log\tan\tfrac{1}{2}\alpha = \log r - \log(s-a),$$

and

$$\log r = \tfrac{1}{2}[\log(s-a) + \log(s-b) + \log(s-c) - \log s].$$

Example. Given $a = 46.34$, $b = 31.27$, $c = 55.03$, to find the angles.

$$\begin{aligned} a &= 46.34 \\ b &= 31.27 \\ c &= 55.03 \\ \hline 2s &= 132.64 \\ s &= 66.32 \end{aligned} \qquad \begin{aligned} s-a &= 19.98 \\ s-b &= 35.05 \\ s-c &= 11.29 \\ \hline & \,66.32 = s \end{aligned}$$

(Notice that $(s-a)+(s-b)+(s-c)=s$, necessarily. Why so?)

To find $\log r$:

$$\log r = \tfrac{1}{2}[\log(s-a) + \log(s-b) + \log(s-c) - \log s].$$

$$\begin{aligned} \log(s-a) &= 1.3006 \\ \log(s-b) &= 1.5447 \\ \log(s-c) &= 1.0527 \\ \hline &\;3.8980 \\ \log s &= 1.8216 \\ \hline \log r^2 &= 2.0764 \\ \log r &= 1.0382 \end{aligned}$$

Therefore

$$\begin{aligned} \log\tan\tfrac{1}{2}\alpha &= 9.7376 - 10 \\ \log\tan\tfrac{1}{2}\beta &= 9.4935 - 10 \\ \log\tan\tfrac{1}{2}\gamma &= 9.9855 - 10 \end{aligned}$$

Therefore

$$\begin{aligned} \tfrac{1}{2}\alpha &= 28^\circ 39' \\ \tfrac{1}{2}\beta &= 17^\circ 18' \\ \tfrac{1}{2}\gamma &= 44^\circ 3' \\ \hline &\;90^\circ 00' \textit{ Check} \end{aligned}$$

Therefore $\alpha = \underline{\underline{57^\circ 18'}}$, $\beta = \underline{\underline{34^\circ 36'}}$, $\gamma = \underline{\underline{88^\circ 6'}}$.

159. If the area is to be found, we may use the formula already proved (p. 159): $S = \frac{1}{2} bc \sin \alpha.$

Now $$\sin \alpha = 2 \sin \frac{\alpha}{2} \cos \frac{\alpha}{2};$$
or, using (1′) and (2) (p. 198),
$$\sin \alpha = \frac{2}{bc}\sqrt{s(s-a)(s-b)(s-c)}.$$

Therefore $$S = \sqrt{s(s-a)(s-b)(s-c)}.$$

This important result is called *Hero's Formula*.[1] Notice that $S = r \cdot s$ also.

EXERCISES

1. Solve and check, and find the area:

(1) $a = 13,\quad b = 14,\quad c = 15.$
(2) $a = .00365,\quad b = .00846,\quad c = .00697.$
(3) $a = 100.1,\quad b = 102.1,\quad c = 104.1.$
(4) $a = 3.194,\quad b = 5.235,\quad c = 5.118.$

2. Prove that $r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}}$ is the radius of the inscribed circle.

160. The only remaining case of the solution of a triangle that requires new formulas for logarithmic work is that in which two sides and the included angle are given. Suppose the given parts are a, b, γ, and $a > b$. Construct the triangle ABC (Fig. 120), and locate the point D on CB, so that $CD = b$; then $BD = a - b$. Produce BC to E, making $CE = b$; then $BE = a + b$.

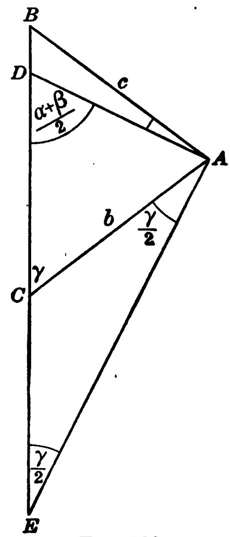


Fig. 120

The student may show that
$$\angle AEC = \angle CAE = \frac{\gamma}{2},$$
and that $$\angle ADC = \frac{\alpha+\beta}{2}.$$

Therefore $$\angle BAD = \alpha - \frac{\alpha+\beta}{2} = \frac{\alpha-\beta}{2}.$$
Also $$\angle DAE = 90°.$$

[1] From Hero of Alexandria, who was the first to prove it (about the first century A.D.). His method was a purely geometric one.

Therefore $\angle BAE = 90° + \frac{\alpha - \beta}{2}.$

Now apply the Law of Sines, first to $\triangle BDA$ and then to $\triangle BEA$.

We get
$$\frac{a-b}{c} = \frac{\sin\frac{\alpha-\beta}{2}}{\sin\frac{\alpha+\beta}{2}} = \frac{\sin\frac{\alpha-\beta}{2}}{\cos\frac{\gamma}{2}}, \tag{1}$$

and
$$\frac{a+b}{c} = \frac{\sin\left(90° + \frac{\alpha-\beta}{2}\right)}{\sin\frac{\gamma}{2}} = \frac{\cos\frac{\alpha-\beta}{2}}{\sin\frac{\gamma}{2}}. \tag{2}$$

Formulas (1) and (2) (leaving out the second fraction in each line) are called Mollweide's Formulas.[1] By dividing (1) by (2) we get

$$\frac{a-b}{a+b} = \frac{\tan\frac{\alpha-\beta}{2}}{\tan\frac{\alpha+\beta}{2}} = \frac{\tan\frac{\alpha-\beta}{2}}{\cot\frac{\gamma}{2}}. \tag{3}$$

This important result is known as the *Law of Tangents*; it provides an easy way of solving by logarithms in case two sides and the included angle are given.

Example. Given $a = 26.71$, $b = 20.89$, $\gamma = 61° 32'$, to find α, β, and c.

$$a + b = 47.60,$$
$$a - b = 5.82,$$
$$\frac{\alpha+\beta}{2} = 90° - \frac{\gamma}{2} = 90° - 30° 46' = 59° 14'.$$

From the Law of Tangents,

$$\log\tan\frac{\alpha-\beta}{2} = \log(a-b) + \log\tan\frac{\alpha+\beta}{2} - \log(a+b).$$

$$\log(a-b) = 0.7649$$
$$\log\tan\frac{\alpha+\beta}{2} = \frac{0.2252}{0.9901}$$
$$\log(a+b) = \underline{1.6776}$$

Therefore
$$\log\tan\frac{\alpha-\beta}{2} = 9.3125 - 10$$

and
$$\frac{\alpha-\beta}{2} = 11° 36'$$

[1] After C. B. Mollweide (1774–1825), who, however, was not the first to discover them. Formula (2) was first given by Newton in 1707, and Formula (1) by Friedrich Wilhelm von Oppel in 1746.

Therefore $$\alpha = \frac{\alpha+\beta}{2} + \frac{\alpha-\beta}{2} = \underline{\underline{70^\circ\,50'}}$$

$$\beta = \frac{\alpha+\beta}{2} - \frac{\alpha-\beta}{2} = \underline{\underline{47^\circ\,38'}}$$

Check. $\alpha + \beta + \gamma = 180^\circ$.

To find c, use either of Mollweide's Formulas; thus (1) gives

$$\log c = \log(a-b) + \log\cos\frac{\gamma}{2} - \log\sin\frac{\alpha-\beta}{2}.$$

$$\begin{aligned}
\log(a-b) &= 0.7649\\
\log\cos\frac{\gamma}{2} &= \underline{9.9341-10}\\
&\ 10.6990-10\\
\log\sin\frac{\alpha-\beta}{2} &= \underline{9.3034-10}\\
\log c &= 1.3956\\
c &= \underline{\underline{24.86}}
\end{aligned}$$

Check. $$c = \frac{(a+b)\sin\frac{\gamma}{2}}{\cos\frac{\alpha-\beta}{2}}$$

$$\begin{aligned}
\log(a+b) &= 1.6776\\
\log\sin\frac{\gamma}{2} &= \underline{9.7089-10}\\
&\ 11.3865-10\\
\log\cos\frac{\alpha-\beta}{2} &= \underline{9.9910-10}\\
\log c &= 1.3955
\end{aligned}$$

EXERCISES

Solve, check, and find the area:

(1) $a = 75,\quad b = 70,\quad \gamma = 56^\circ\,18'$.

(2) $a = 463,\quad b = 499,\quad \gamma = 28^\circ\,6'$.

(3) $b = 2.68,\quad c = 1.59,\quad \alpha = 80^\circ\,54'$.

(4) $a = 41.3,\quad b = 28.7,\quad \gamma = 136^\circ\,28'$.

NOTE. Inasmuch as many of the formulas used have not been proved for the case $\alpha > 90^\circ$, the student should supply this omission here.

161. These examples illustrate but a few of the very large number of applications of logarithmic computations. The subjects of surveying, navigation, and astronomy make constant use of this labor-saving device. It may, indeed, be said to have revolutionized numerical work in every field.

We now return to the study of the logarithmic function itself. The best way to get a general idea of this function (as of any other) is by drawing a careful graph of it. Accordingly, writing $y = \log_{10} x$, we lay off x-values as abscissas and y-values as ordinates (choosing a suitable scale, as usual). As we have had no definition of a logarithm of a *negative* number, no points can be located to the left of the y-axis. The details of forming the graph are left to the student.

CHAPTER XI

INTRODUCTION TO THE DIFFERENTIAL CALCULUS

162. The preceding chapters have been devoted to a study of various types of functions, — their fundamental properties, their graphical representation, and some algebraic and geometric results derived therefrom. The linear function, the quadratic function, the fractional function, the irrational function, the trigonometric functions, the logarithmic and exponential functions, all have their individual characteristics, which are in each case intimately connected with their graphs, and which must be well understood as a preliminary to the solving of many important problems. A complete understanding of a function would involve a complete knowledge of the way in which the function changes as the independent variable changes. The object of this chapter is to lay the foundation for such an understanding. The branch of mathematics which investigates the *rate of change of a function* is called the *differential calculus.*

163. Although this branch of mathematics is of vast complexity and difficulty in its advanced aspects (indeed, no one can even faintly imagine the extent which future discoveries may give to its applications), yet in its simpler aspects it is exceedingly simple. This will be seen by examining one of the simple functional relations, which has already been mentioned (p. 23), — the case of a man walking at the constant rate of 3 mi. per hour. If t represents the number of hours he has walked, and s the number of miles he has gone, then s is a function of $t\,[s = f(t)]$. Evidently the functional relation is the linear one $s = 3\,t$. If we consider two different values of t, say $t = 2$ and $t = 2\frac{1}{4}$, the corresponding values of s are 6 and $6\frac{3}{4}$; that is, the additional distance he has gone in the extra $\frac{1}{4}$ hr. is $\frac{3}{4}$ mi. Again, if we take $t = 3$ and $t = 3\frac{1}{4}$, we find that the additional distance he has gone in the extra $\frac{1}{4}$ hr. is

again $\frac{3}{4}$ mi. In fact, it is evident that *any* $\frac{1}{4}$-hr. increase in the time spent will produce an increase of $\frac{3}{4}$ mi. in the distance covered; and that is exactly what is involved in the statement that his *rate* is constant.

164. Contrast with this the functional relation between distance and time that exists in the case of a body falling freely near the surface of the earth. The Law of Falling Bodies[1] gives this functional relation to be (neglecting the resistance of the air) $s = 16\,t^2$, where t is the number of seconds the body has been falling and s the number of feet it has fallen. If we give t two different values, say $t = 2$ and $t = 2\frac{1}{4}$, we find the corresponding values of s to be 64 and 81 (that is, the additional fall during the extra $\frac{1}{4}$ sec. is 17 ft.); but if we take $t = 3$ and $t = 3\frac{1}{4}$, we find the corresponding values of s to be 144 and 169, the additional fall during the extra $\frac{1}{4}$ sec. being thus 25 ft. In fact, we shall find that an increase of $\frac{1}{4}$ in t will produce a *different* amount of change in s, according to the value of t chosen to begin with. This is what is involved in the statement that the rate of fall is *not* constant.

165. These two simple examples contain the essential material out of which the structure of the differential calculus is built; namely, the ideas of *functional relation* and *rate of change of a function*. A number of examples of a similar kind will now be considered, and although the computations involved are, as was said above, extremely simple, still they bring out the essential thing, which is, the effect upon a function of changing the independent variable by a certain amount. This "amount of change" is called an *increment*, and is to be considered positive if the change is an *increase*, negative if the change is a *decrease*. Increments are usually symbolized by the Greek letter Δ prefixed to the letter standing for the variable in question. Thus, "the increment of t" means "the amount of change in t" and is symbolized by Δt. In the examples above, $\Delta t = \frac{1}{4}$ (that is, $\frac{1}{4}$ hr. in the first example and $\frac{1}{4}$ sec. in the second example). As a further abbreviation the symbol

[1] Discovered and proved by Galileo (1564–1642); published in his "Dialogues on Two New Sciences," in 1638.

$f(2)$ is used for "the value of $f(t)$ when $t = 2$" or "the value of $f(x)$ when $x = 2$." Thus, if $f(t) = 16\,t^2$, then $f(2) = 16 \cdot 2^2 = 64$, $f(0) = 16 \cdot 0^2 = 0$, $f(a) = 16 \cdot a^2$, $f(b+1) = 16(b+1)^2$, etc.

Example 1. Take the function $s = 5\,t + 6$, and find the effect upon the value of s of an increase of 1 in the value of t,[1] (a) when $t = 2$, (b) when $t = 4$, (c) when $t = \frac{1}{2}$, (d) when $t = t_1$.

Solution. $s = f(t) = 5\,t + 6.$

(a) When $t = 2,\ s = f(2) = 5 \cdot 2 + 6 = 16.$

When $t = 2 + 1,\ s = f(3) = 5 \cdot 3 + 6 = 21.$

Therefore the change in s, Δs, is equal to $f(3) - f(2) = 21 - 16 = 5.$

(b) When $t = 4,\ s = f(4) = 5 \cdot 4 + 6 = 26.$

When $t = 4 + 1,\ s = f(5) = 5 \cdot 5 + 6 = 31.$

Therefore the change in s, Δs, is equal to $f(5) - f(4) = 31 - 26 = 5.$

(c) When $t = \frac{1}{2},\ s = f(\frac{1}{2}) = 5 \cdot \frac{1}{2} + 6 = 8\frac{1}{2}.$

When $t = \frac{1}{2} + 1,\ s = f(\frac{3}{2}) = 5 \cdot \frac{3}{2} + 6 = 13\frac{1}{2}.$

Therefore $\Delta s = f(\frac{3}{2}) - f(\frac{1}{2}) = 13\frac{1}{2} - 8\frac{1}{2} = 5.$

(d) When $t = t_1,\ s = f(t_1) = 5\,t_1 + 6.$

When $t = t_1 + 1,\ s = f(t_1 + 1) = 5(t_1 + 1) + 6 = 5\,t_1 + 11.$

Therefore $\Delta s = f(t_1 + 1) - f(t_1) = 5.$

This result shows, since t_1 is *any* value of t, that $\Delta s = 5$ for *all values* of t when $\Delta t = 1$; that is, a change of 1 in t produces a change of 5 in s, no matter what value of t we start with. In the same way it can be shown that a change of any amount Δt in t produces a change of $5\,\Delta t$ in s, whatever value of t be chosen to begin with. This means that the *rate of change* of s, compared with that of t, is constant, s changing five times as much as t does ($\Delta s = 5\,\Delta t$).

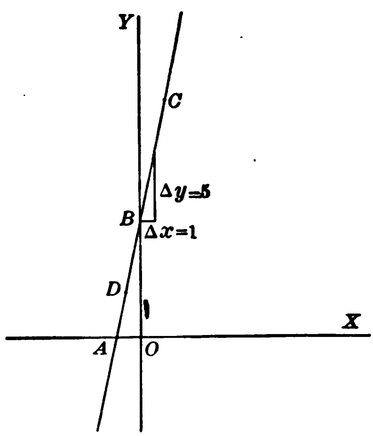


FIG. 121

The graphical representation of the function s will throw more light upon these results. If we use x and y to stand for the variables, instead of s and t, the functional relation will be $y = 5\,x + 6$. The graph of this is the straight line with slope 5 and Y-intercept 6. The work above proved that for *any* value of x, $\Delta y = 5$

[1] In symbols this would be stated, Find the value of Δs when $\Delta t = 1$.

when $\Delta x = 1$. This means that, whatever point on the straight line we start with, whether A, B, C, D, or any other, an increase of 1 in x produces an increase of 5 in y. Geometrically this is evident, because of the fact that the graph is a straight line. (The student may prove by elementary geometry that this is true.) Show further that the function $y = mx + k$ gives always $\Delta y = m \cdot \Delta x$, whatever value of x be chosen to begin with.

Example 2. Let us now consider a quadratic function, choosing $y = x^2$ as the simplest one. Starting with any value of x we please, say $x = x_1$, let us find out what change in the value of y will be produced by a change Δx in the value of x. When $x = x_1$, $y = f(x_1) = x_1^2$.

When $\quad x = x_1 + \Delta x,\ y = f(x_1 + \Delta x) = (x_1 + \Delta x)^2.$

Therefore $\quad \Delta y = f(x_1 + \Delta x) - f(x_1) = (x_1 + \Delta x)^2 - x_1^2 = 2\,x_1 \Delta x + (\Delta x)^2.$

This is evidently not the same for all values of x_1, so that the rate of change of this function is not constant. For instance, if $x_1 = 1$, $\Delta y = 2\,\Delta x + (\Delta x)^2$; while if $x_1 = \frac{1}{2}$, $\Delta y = \Delta x + (\Delta x)^2$, and if $x_1 = 3$, $\Delta y = 6\,\Delta x + (\Delta x)^2$. Reference to the graph will make this fact still clearer. Thus, in Fig. 122, if $P \equiv (x_1, y_1)$ and if $PR = \Delta x$, then $RQ = \Delta y$, $SP = y_1 = f(x_1)$, $TQ = f(x_1 + \Delta x) = y_1 + \Delta y$. It is geometrically evident that for the same value of PR the length of RQ will be different according to where P is located upon the curve. This of course agrees with the result of the algebraic work above, which showed that $RQ = 2\,x_1 \Delta x + (\Delta x)^2$, a length which is different according to the value of x_1 chosen.

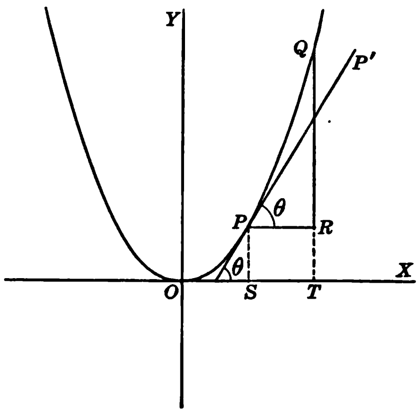


FIG. 122

EXERCISES

Work through the same study as in Examples 1 and 2 for each of the following functions, illustrating by the graph:

1. $s = 4\,t + 2$.	4. $s = \frac{1}{2}\,t - \frac{3}{5}$.	7. $y = 2\,x^2 - 1$.
2. $s = -\,2\,t + 1$.	5. $y = -\,5\,x + 3$.	8. $y = 1 - x^2$.
3. $y = 10\,x + 25$.	6. $y = x^2 + 3$.	9. $y = 2 - 3\,x^2$.

In particular, find the value of Δy (or Δs) when x (or t) $= 1$ and Δx (or Δt) $= 1$, .1, and .01 respectively.

166. Instantaneous rate of change of a function at a point. Returning to the function $y = x^2$ of Example 2, § 165, we had established the fact that for any given value of x, such as $x = x_1$, $\Delta y = 2\,x_1 \cdot \Delta x + (\Delta x)^2$. This is the *total change in y* produced by increasing x by the amount Δx. Hence the *average* change in y *per unit* of Δx will be found by dividing Δy by Δx,[1] the average rate of change $\frac{\Delta y}{\Delta x}$, or $2\,x_1 + \Delta x$. If now we take Δx smaller and smaller, we get the average rate of change in y throughout an ever-diminishing interval, and this *average rate* will in this case (and generally) approach a definite value, here $2\,x_1$, as Δx approaches 0. This value, which $\frac{\Delta y}{\Delta x}$ approaches as Δx diminishes, is called the *instantaneous rate of change of y relative to x* at the point $x = x_1$. This instantaneous rate of change is what we mean when we speak of the rate of change of any variable at a definite point or at a definite time. Thus, to state that a train is at a certain moment moving at the rate of 30 mi. per hour does not necessarily mean that it continues to move at that rate for an hour, or even for a single second; it means, rather, that the *instantaneous rate* (namely, the *limiting value* of the average rate throughout an interval of time or of space, as the interval is conceived to diminish toward zero) is 30 mi. per hour. Of course it would be impossible by any process of *observation* or experiment to determine what this limiting value is; it is determined by a process of *reasoning*, which may be very simple or very difficult, according to the nature of the functional relation involved. Only in case the variable is changing at a *constant* rate, as in Ex. 1, p. 205, can we avoid this consideration of a limiting value, because only then can we take *any* fixed interval we please and be sure that the rate of change for this interval is the same as that for any other interval. In that example (p. 205) we saw that $\Delta s = 5\,\Delta t$; that is, $\frac{\Delta s}{\Delta t} = 5$, a *constant*, so that the rate of change of s is always 5 times the rate of change of t. But if the rate is not constant, we cannot

[1] If the total change in y were 10 for $\Delta x = 3$, the *average* change in y per unit of Δx would be $\frac{10}{3}$.

do this, but must consider what happens to the ratio $\frac{\Delta s}{\Delta t}\left(\text{or } \frac{\Delta y}{\Delta x}\right)$ as the interval Δt (or Δx) becomes smaller and smaller. It is thus evident that a study of the rate of change of a variable will necessarily involve a study of *limiting values*, or, more briefly, *limits*.

167. Limits. The student has already met with a few examples of limits; for instance, in elementary geometry it is stated that the limit of the perimeter of a regular polygon inscribed in a circle, as the number of sides is indefinitely increased, is the circumference of the circle, and that the limit of the apothem of the polygon is the radius of the circle. The following are other examples of limits.

1. If x represents a variable which assumes the series of values $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \cdots$, then x approaches the limit 0; in symbols, $\lim x = 0$, or $x \doteq 0$.

2. If x takes on the series of values .9, .99, .999, $\cdots$, then $\lim x = 1$, or $x \doteq 1$.

3. If $y = \frac{1}{x}$ and $x \doteq 10$, then $y \doteq \frac{1}{10}$; this statement is abbreviated thus:

$$\lim_{x \doteq 10} y = \frac{1}{10}, \quad \text{or} \quad \lim_{x \doteq 10}\left(\frac{1}{x}\right) = \frac{1}{10}.$$

4. If $y = 2 + h$ and $h \doteq 0$, then $y \doteq 2$; in symbols, $\lim_{h \doteq 0}(2+h) = 2$.

5. If $y = \frac{1}{x}$ and x is *positive* and diminishes toward 0 as limit, y will *increase* without limit; in symbols, $\lim_{x \doteq 0^+} y = \infty$, or $\lim_{x \doteq 0^+}\left(\frac{1}{x}\right) = \infty$.

6. If $y = \frac{1}{x}$ and x is *negative* and approaches 0 as limit, y will *decrease* without limit (since it is *negative* and its numerical value becomes larger and larger); in symbols, $\lim_{x \doteq 0^-} y = -\infty$, or $\lim_{x \doteq 0^-}\left(\frac{1}{x}\right) = -\infty$.

7. If $y = \frac{1}{2^n}$, where $n = 1, 2, 3, 4, \cdots$, y approaches 0 as limit; in symbols, $\lim_{n \doteq \infty}\left(\frac{1}{2^n}\right) = 0$.

8. If $y = \frac{1}{2^n}$, where $n = -1, -2, -3, -4, \cdots$, y *increases* without limit; in symbols, $\lim_{n \doteq -\infty}\left(\frac{1}{2^n}\right) = \infty$.

9. If $x = 1 + (-\frac{1}{2})^n$ where $n = 1, 2, 3, 4, \cdots$, $\lim\limits_{n \doteq \infty} x = 1$.

10. If $x = \dfrac{2+z}{3-z}$ and z increases without limit, x approaches -1 as limit, that is, $\lim\limits_{z \doteq \infty} x = -1$; if $z \doteq 0$, $\lim x = \frac{2}{3}$.

11. If $y = \sin x$ and $x \doteq 90°$, $y \doteq 1$; that is, $\lim\limits_{x \doteq 90} y = 1$.

12. If $y = \sin\dfrac{1}{x}$ and $x \doteq 0$, y *has no limit*, because $\dfrac{1}{x}$ increases without limit, and hence $\sin\dfrac{1}{x}$ will take on all values between -1 and $+1$ over and over again; that is, the values of y will *oscillate* as x diminishes.

13. If $y = \log x$, $\lim\limits_{x \doteq 1} y = 0$.

14. If PQ is the straight line joining any two points P and Q on a curve, and if Q approaches P along the curve, the straight line PQ will usually approach a limiting position PT, which is called the *tangent* to the curve at P. Also, $\angle BCQ \doteq \angle BDT$ (Fig. 123).

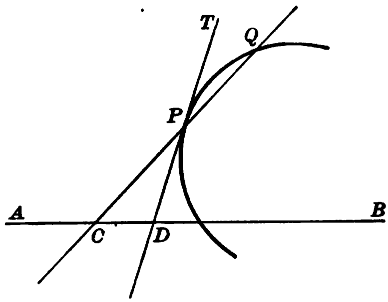


FIG. 123

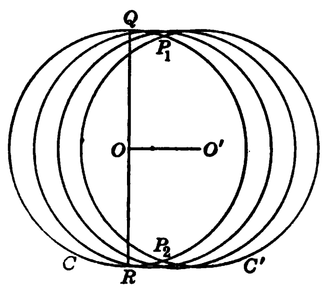


FIG. 124

15. If two circles C and C', of equal radii, intersect, and if C' approaches C, its center O' remaining on the straight line $O'O$, then the points of intersection P_1 and P_2 will approach Q and R respectively, where Q and R are the ends of the diameter through O, perpendicular to $O'O$ (Fig. 124).

168. Definition of limit. These examples make the idea of limit sufficiently clear so that we are prepared for a *definition* of the word: A variable v approaches a constant l as a limit if

$|v - l|$ becomes[1] and remains less than any assignable positive number ϵ (epsilon). The above thirteen algebraic examples should be thought through carefully to verify that this definition is satisfied in each case. In the geometric examples (Exs. 14 and 15) it is either an *angle* or a *length* that is the variable, so that here also the definition will be seen to be satisfied.

EXERCISES

Show that the following limit equations are true:

1. If $v = 1 + \frac{1}{n}$, where $n = 1, 2, 3, 4, \cdots$, then $v \doteq 1$.

2. If $v = \frac{1+x}{2-x}$, $\lim\limits_{x \doteq 0} v = \frac{1}{2}$.

3. $\lim\limits_{x \doteq 0} \frac{1}{1-x} = 1.$

4. $\lim\limits_{x = \infty} \frac{1}{1-x} = 0.$

5. $\lim\limits_{x = \infty} \frac{x}{1+x} = 1.$

6. $\lim\limits_{x = 180^\circ} [\tan^2(x - \theta) - \cos x] = \sec^2\theta.$

7. $\lim\limits_{x \doteq 2} \frac{x^2 - 1}{x - 1} = 3$; $\lim\limits_{x \doteq 1} \frac{x^2 - 1}{x - 1} = 2.$

8. $\lim\limits_{x \doteq 1} \frac{x^3 - 1}{x - 1} = 3$; $\lim\limits_{x \doteq 1} \frac{x^n - 1}{x - 1} = n$, n being a positive integer.

9. $\lim\limits_{n \doteq \infty} \frac{n(n+1)}{(n+2)(n+3)} = 1.$

169. Theorems on limits. The following theorems are easily seen to result from the definition of a limit:

I. If $\lim v = l$, then $\lim (v + c) = l + c$, c being any constant.

II. If $\lim v = l$, then $\lim (cv) = cl$, c being any constant.

III. If $\lim v_1 = l_1$, and $\lim v_2 = l_2$, then $\lim (v_1 \pm v_2) = l_1 \pm l_2$.

IV. If $\lim v_1 = l_1$, and $\lim v_2 = l_2$, then $\lim (v_1 v_2) = l_1 l_2$.

V. If $\lim v_1 = l_1$, and $\lim v_2 = l_2$, then

$$\lim\left(\frac{v_1}{v_2}\right) = \frac{l_1}{l_2} \text{ (\textit{provided} } l_2 \neq 0).$$

[1] $|v - l|$ means "the numerical value of $v - l$"; that is, $v - l$ itself if $v > l$, but $l - v$ if $v < l$.

170. The derivative. The most important case of a limiting value with which we have to do is that of the instantaneous rate of change of a function, described in § 166. It is the limit of the quotient $\frac{\Delta y}{\Delta x}$ as Δx approaches 0. In the case of the function $y = x^2$ we found that $\frac{\Delta y}{\Delta x} = 2x_1 + \Delta x$. Hence for this function we have the fact that

$$\lim_{\Delta x \doteq 0} \frac{\Delta y}{\Delta x} = 2x_1.$$

This limiting value of the quotient $\frac{\Delta y}{\Delta x}$ is called the *derivative of y with respect to x at the point* $x = x_1$. In words, the derivative is the limit of the ratio

$$\frac{\text{increment of the function}}{\text{increment of the independent variable}}$$

as the increment of the independent variable approaches 0. The symbol used is $D_x y]_{x=x_1}$, read "the derivative of y with respect to x at the point $x = x_1$."

Thus $$D_x y]_{x=x_1} = \lim_{\Delta x \doteq 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \doteq 0} \frac{f(x_1 + \Delta x) - f(x_1)}{\Delta x}.$$

If the function is $s = f(t)$, the corresponding statement will be

$$D_t s]_{t=t_1} = \lim_{\Delta t \doteq 0} \frac{\Delta s}{\Delta t} = \lim_{\Delta t \doteq 0} \frac{f(t_1 + \Delta t) - f(t_1)}{\Delta t}.$$

Other symbols often used for the derivative of $f(x)$ with respect to x are $D_x f(x)$ and $f'(x)$. The last form is especially convenient if we wish to show clearly for what value of x the derivative is to be understood; for example, $f'(x_1)$, $f'(0)$, etc. mean "the derivative at the point $x = x_1$, $x = 0$," etc.

171. The derivative as the slope of the tangent to a curve. We have seen that the derivative of a function gives the value of the instantaneous rate of change of the function in relation to that of the independent variable. It can also be interpreted in another way, as can be seen by referring to Fig. 122, p. 206, which is the graph

of the function $y = x^2$. In this figure $RQ = \Delta y = f(x_1 + \Delta x) - f(x_1) = 2x_1\Delta x + (\Delta x)^2$, and $PR = \Delta x$. The quotient $\frac{\Delta y}{\Delta x} = \frac{RQ}{PR}$ is the value of $\tan \angle RPQ$; that is, it is the *slope of the line* PQ. If we now let Δx diminish toward zero, the point Q will approach the point P along the curve, and the chord PQ will approach as limiting position the line PP', which is the *tangent* to the curve at P. Also, the angle RPQ will approach as limit the angle $RPP' = \theta$, and therefore $\tan \angle RPQ \doteq \tan\theta$; that is,

$$\lim_{\Delta x \doteq 0} \frac{\Delta y}{\Delta x} = \tan\theta.$$

In words, *the limiting value of* $\frac{\Delta y}{\Delta x}$ *as* Δx *approaches 0 is the slope of the tangent to the curve at the point* $x = x_1$; or, *the derivative of* y *with respect to* x *gives the slope of the tangent to the curve at the point for which the derivative was formed.*

Thus, the derivative of the function $y = x^2$ being $2x_1$ at any point $x = x_1$ (§ 170) we can say that the slope of the tangent to the curve $y = x^2$ at the point $x = x_1$ is $2x_1$.

172. We shall now consider by means of some examples the method of finding the derivatives of a few functions.

Example 1. Find the derivative of the function $y = 2x^2 - x$ at any point $x = x_1$.

Solution. Let y_1 be the value of y when $x = x_1$. Then $y_1 = f(x_1) = 2x_1^2 - x_1$,

and $\qquad y_1 + \Delta y = f(x_1 + \Delta x) = 2(x_1 + \Delta x)^2 - (x_1 + \Delta x).$

Hence, by subtraction,

$$\Delta y = 4x_1\Delta x + 2(\Delta x)^2 - \Delta x$$

and

$$\frac{\Delta y}{\Delta x} = 4x_1 - 1 + 2\Delta x.$$

Therefore

$$\lim_{\Delta x \doteq 0} \frac{\Delta y}{\Delta x} = 4x_1 - 1;$$

that is,

$$D_x y]_{x=x_1} = 4x_1 - 1.$$

Thus the slope of the tangent to the curve $y = 2x^2 - x$ at the point (x_1, y_1) is $4x_1 - 1$. Draw a figure.

Example 2. Find the derivative of the function $y = \frac{1}{x}$.

Solution. From now on we shall drop the subscripts and write, for the point at which the derivative is to be found, (x, y) instead of (x_1, y_1). Then $y = f(x) = \frac{1}{x}$.

$$y + \Delta y = f(x + \Delta x) = \frac{1}{x + \Delta x}.$$

Therefore $$\Delta y = \frac{1}{x + \Delta x} - \frac{1}{x} = \frac{-\Delta x}{x(x + \Delta x)},$$

and $$\frac{\Delta y}{\Delta x} = \frac{-1}{x(x + \Delta x)}.$$

Hence $$\lim_{\Delta x \doteq 0} \frac{\Delta y}{\Delta x} = -\frac{1}{x^2} \text{ (Theorem V, § 169), provided } x \neq 0;$$

that is, $D_x\left(\frac{1}{x}\right) = -\frac{1}{x^2}$, or the slope of the tangent to the curve $y = \frac{1}{x}$ at the point (x, y) is $-\frac{1}{x^2}$. Draw a figure.

Example 3. Find the derivative of $\frac{x-1}{2x+3}$ at any point (x, y).

Solution. $$y = \frac{x-1}{2x+3},$$

$$y + \Delta y = \frac{(x + \Delta x) - 1}{2(x + \Delta x) + 3}.$$

Therefore

$$\Delta y = \frac{x + \Delta x - 1}{2(x + \Delta x) + 3} - \frac{x-1}{2x+3}$$

$$= \frac{(2x+3)(x-1) + (2x+3)\Delta x - (2x+3)(x-1) - 2\Delta x(x-1)}{(2x+3)(2x+3+2\Delta x)}$$

$$= \frac{5\Delta x}{(2x+3)(2x+3+2\Delta x)},$$

$$\frac{\Delta y}{\Delta x} = \frac{5}{(2x+3)(2x+3+2\Delta x)}.$$

Therefore, $$\lim_{\Delta x \doteq 0} \frac{\Delta y}{\Delta x} = \frac{5}{(2x+3)^2};$$

that is, $$D_x\left(\frac{x-1}{2x+3}\right) = \frac{5}{(2x+3)^2}.$$

The student should verify this result in a figure, using several special values for x, such as $x = 1$, $x = 0$, $x = -1$.

EXERCISES

Find the derivatives of each of the following functions, and draw figures:

1. $y = x^2 + 5x$.
2. $y = x^3$.
3. $y = 3x^2 - 4x$.
4. $y = x^2 - 3x + 1$.
5. $y = 2x^2 + 3x - 2$.
6. $y = \dfrac{1}{x+1}$.
7. $y = \dfrac{2x+3}{x-1}$.
8. $y = \dfrac{x}{1+x}$.
9. $y = \dfrac{x}{1+x^2}$.
10. $y = \dfrac{x-2}{x-3}$.
11. $y = \dfrac{1}{5x-6}$.
12. $y = \dfrac{3}{1-x^2}$.
13. $y = \dfrac{2x}{1+x^2}$.
14. $y = \dfrac{x}{(1+x^2)}$.

15. Find the derivatives of x^2+1, x^2-1, x^2-3, and x^2+k. Compare with the value of $D_x(x^2)$.

173. Rules for finding derivatives. The derivatives of all the algebraic functions that we have studied in this book can be found[1] by methods like those illustrated by the above examples. In practice these computations can be greatly shortened by breaking up more complicated functions into simple parts, according to the following rules:

I. *The derivative of a constant is zero.* $D_x c = 0$.

For a constant does not change its value at all; hence if $y = c$, $\Delta y = 0$ for all values of Δx. Therefore $\lim\limits_{\Delta x \doteq 0} \dfrac{\Delta y}{\Delta x} = 0$, which proves the theorem.

II. *The derivative of a variable with respect to itself is unity.* $D_x x = 1$.

For if $y = x$, $\Delta y = \Delta x$. Therefore $\dfrac{\Delta y}{\Delta x} = 1$ for all values of Δx, and hence $\lim\limits_{\Delta x \doteq 0} \dfrac{\Delta y}{\Delta x} = 1$.

III. *If u and v are two functions of x, each having derivatives $D_x u$ and $D_x v$, then the function $u + v$ has as its derivative $D_x u + D_x v$.* $D_x(u+v) = D_x u + D_x v$.

[1] That is, they can be found at every point for which they exist. A function may have a value for some values of the independent variable for which nevertheless no derivative exists, — for example, $\sqrt{x}$ for $x = 0$.

For if $y = u + v$, and if the increments of y, u, and v, caused by giving x the increment Δx, are represented by Δy Δu, and Δv respectively, then

$$y + \Delta y = (u + \Delta u) + (v + \Delta v);$$

therefore

$$\Delta y = \Delta u + \Delta v,$$

$$\frac{\Delta y}{\Delta x} = \frac{\Delta u}{\Delta x} + \frac{\Delta v}{\Delta x}.$$

By hypothesis,

$$\lim_{\Delta x \doteq 0} \frac{\Delta u}{\Delta x} = D_x u \quad \text{and} \quad \lim_{\Delta x \doteq 0} \frac{\Delta v}{\Delta x} = D_x v.$$

Hence, by Theorem III, § 169,

$$\lim_{\Delta x \doteq 0} \frac{\Delta y}{\Delta x} = D_x u + D_x v. \qquad \text{Q.E.D.}$$

Evidently this result can be extended to the sum of any number of functions. Stated in words it is as follows:

The derivative of the sum of any number of functions whose derivatives exist equals the sum of their derivatives.

Example 1. We have found that $D_x(x^2) = 2x$, and that $D_x(x^3) = 3x^2$ (Exercise 2, § 172). Therefore $D_x(x^3 + x^2) = 3x^2 + 2x$, so that it is not necessary to work out the computation for the function $x^3 + x^2$ anew.

Example 2. $D_x(x^3) = 3x^2$, (Exercise 2, § 172)

and $D_x(x^2 - 3x + 1) = 2x - 3$. (Exercise 4, § 172)

Therefore $D_x(x^3 + x^2 - 3x + 1) = 3x^2 + 2x - 3$.

IV. *If u and v are two functions of x, having derivatives $D_x u$ and $D_x v$, then the function $u \cdot v$ has as its derivative $uD_x v + vD_x u$;* that is, $D_x(uv) = uD_x v + vD_x u$.

Proof. If $y = uv$, and Δy, Δu, and Δv have the usual meaning, then

$$y + \Delta y = (u + \Delta u)(v + \Delta v) = uv + u\Delta v + v\Delta u + \Delta u \Delta v.$$

Therefore

$$\Delta y = u\Delta v + v\Delta u + \Delta u \Delta v,$$

and

$$\frac{\Delta y}{\Delta x} = u\frac{\Delta v}{\Delta x} + v\frac{\Delta u}{\Delta x} + \frac{\Delta u}{\Delta x} \cdot \Delta v.$$

By hypothesis, $\lim\limits_{\Delta x \doteq 0} \dfrac{\Delta v}{\Delta x} = D_x v$, $\lim\limits_{\Delta x \doteq 0} \dfrac{\Delta u}{\Delta x} = D_x u$, and, by Theorem IV, § 169, $\lim\limits_{\Delta x \doteq 0} \dfrac{\Delta u}{\Delta x} \cdot \Delta v = 0$, since the limit of the first factor is $D_x u$ and the limit of the second factor is 0.

Therefore $$\lim_{\Delta x \doteq 0} \frac{\Delta y}{\Delta x} = uD_x v + vD_x u. \qquad \text{Q.E.D.}$$

In words, *The derivative of the product of two functions is equal to the first times the derivative of the second, plus the second times the derivative of the first.*

Evidently this rule can be extended to a product of any number of functions; thus, if $y = uvw$, $Dy = u \cdot D(vw) + vw \cdot Du = uv \cdot Dw + uw \cdot Dv + vw \cdot Du$, and so on for any number of functions.

Special cases. (1) If $v = c$, any constant, the rule becomes

$$D_x(c \cdot u) = cD_x u + uD_x c.$$

But, by Rule I, $\quad D_x c = 0.$

Therefore $\quad D_x(cu) = cD_x u.$

Example. Since $D_x(x^2) = 2x$, $D_x(5x^2) = 10x$, $D_x(50x^2) = 100x$, etc.

(2) Another special case is $y = u^n$, n being any positive integer. Then, using the extended form of the rule (for n factors), we have $D_x y = u^{n-1}D_x u + u^{n-1}D_x u + u^{n-1}D_x u + \cdots$ (n such terms all precisely the same).

Therefore $\quad D_x y = n \cdot u^{n-1} D_x u.$

Example 1. Find the derivative of x^{10}.

$$D_x(x^{10}) = 10 \cdot x^9 \cdot D_x x.$$

By Rule II, $\quad D_x x = 1.$

Therefore $\quad D_x(x^{10}) = 10x^9.$

In general, we can write down the result for x^n just as easily:

$$D_x(x^n) = n \cdot x^{n-1} D_x x = n \cdot x^{n-1} \ (n \text{ any positive integer}).$$

Example 2. Find the derivative of $(2x^2 + 3x - 2)^5$.

Let $\quad y = (2x^2 + 3x - 2)^5 = u^5,$

where $\quad u = 2x^2 + 3x - 2.$

Then $\quad D_x y = 5 \cdot u^4 D_x u.$

By Ex. 5, § 172, $D_x u = D_x(2x^2 + 3x - 2) = 4x + 3$. (This can also be obtained by direct application of Rule IV(1), Rule III, and Rule I.)

Therefore $D_x(2x^2 + 3x - 2)^5 = 5(2x^2 + 3x - 2)^4(4x + 3)$.

V. $D_x\left(\dfrac{u}{v}\right) = \dfrac{vD_x u - uD_x v}{v^2}$, *u and v being any two functions having derivatives* $D_x u$ *and* $D_x v$.

In words, *The derivative of a fractional function is equal to the denominator times the derivative of the numerator, minus the numerator times the derivative of the denominator, divided by the square of the denominator (this excludes of course points at which the denominator equals zero).*

Proof. Let
$$y = \frac{u}{v}.$$

Then
$$y + \Delta y = \frac{u + \Delta u}{v + \Delta v}.$$

Therefore
$$\Delta y = \frac{u + \Delta u}{v + \Delta v} - \frac{u}{v} = \frac{v\Delta u - u\Delta v}{v(v + \Delta v)},$$

and
$$\frac{\Delta y}{\Delta x} = \frac{v\dfrac{\Delta u}{\Delta x} - u\dfrac{\Delta v}{\Delta x}}{v(v + \Delta v)}.$$

Therefore
$$\lim_{\Delta x \doteq 0} \frac{\Delta y}{\Delta x} = \frac{vD_x u - uD_x v}{v^2}. \quad \text{(Theorems III and V, § 169.)}$$

Q.E.D.

Special case. If $u = c$, any constant, this rule takes the form
$$D_x\left(\frac{c}{v}\right) = \frac{vD_x c - cD_x v}{v^2} = \frac{-cD_x v}{v^2},$$
since
$$D_x c = 0.$$

The case $v = c$ is not to be taken as a special case of this rule, because $\dfrac{u}{c}$ is not really a fractional function at all (cf. § 68, p. 85). Rather it is to be treated as a *product* $\dfrac{1}{c} \cdot u$, giving
$$D_x\left(\frac{u}{c}\right) = \frac{1}{c} \cdot D_x u;$$

for example,
$$D_x\left(\frac{x^2}{5}\right) = \frac{1}{5} D_x x^2 = \frac{2x}{5}.$$

174. By the help of the foregoing rules we can work out, with the minimum expenditure of time and effort, the derivative of any rational function. (The process of finding derivatives is called *differentiation.*)

Example 1. Differentiate the function $3x^4 - 6x^3 + 5x^2 - 1$.

Solution. By Rule III,

$$\begin{aligned} D_x(3x^4 - 6x^3 + 5x^2 - 1) &= D_x(3x^4) - D_x(6x^3) + D_x(5x^2) - D_x \quad (1)\\ &= 3D_x(x^4) - 6D_x(x^3) + 5D_x(x^2) - 0 \quad \text{(Rules IV (1) and I)}\\ &= 3\cdot 4x^3 - 6\cdot 3x^2 + 5\cdot 2x \quad \text{(Rule IV (2))}\\ &= 12x^3 - 18x^2 + 10x. \end{aligned}$$

Of course it is possible to take the intervening steps *mentally,* and the student should observe that we can write down at once

$$D_x(3x^4 - 6x^3 + 5x^2 - 1) = 12x^3 - 18x^2 + 10x.$$

Example 2. Differentiate the function $(4x^2 - 3x)^3$.

Solution. By Rule IV (2),

$$\begin{aligned} D_x(4x^2 - 3x)^3 &= 3(4x^2 - 3x)^2 D_x(4x^2 - 3x)\\ &= 3(4x^2 - 3x)^2(8x - 3). \end{aligned}$$

Here also the intervening step can be taken mentally, enabling us to write at once

$$D_x(4x^2 - 3x)^3 = 3(8x - 3)(4x^2 - 3x)^2.$$

Example 3. Differentiate the function $\frac{x^2}{1 - x^2}$.

Solution.

$$D_x\left(\frac{x^2}{1 - x^2}\right) = \frac{(1 - x^2)2x - x^2(-2x)}{(1 - x^2)^2} = \frac{2x}{(1 - x^2)^2}.$$

EXERCISES

Differentiate each of the following functions (doing as much of the work as possible mentally):

1. $\frac{x}{1 + x^3}$.
2. $\left(\frac{x}{3}\right)^4$.
3. $\frac{1}{x^2}$.
4. $\frac{10}{x^3}$.
5. $\frac{ax + b}{cx + d}$.
6. $\left(\frac{x}{1 + x}\right)^2$.
7. $\frac{1 + x}{1 + x^2}$.
8. $x^7 - 5x^6 + x^3$.
9. $\frac{2x^3 - 3x^2 + 1}{x^2}$.
10. $\frac{1 + x + x^2}{1 + x - x^2}$.
11. $(x^3 - 5x^2 + 1)^3$.
12. $\frac{a^2x^2}{b^2 + c^2x^2}$.
13. Verify the results of Exs. 1–14, § 172, using rules I–V.

175. Use of the derivative in drawing graphs. Since the value of the derivative of y with respect to x (D_xy) gives the slope of the tangent to the curve $y=f(x)$ at any point, the derivative shows the *direction* of the curve at that point, the direction of the curve being that of the tangent line. In particular, if the derivative is *positive*, the tangent line makes an *acute* angle with the X-axis, and the curve is *rising* as we go along it toward the right (Fig. 125). If the derivative is *negative*, the tangent line makes an *obtuse* angle with the X-axis, so that the curve is *falling* toward the right (Fig. 126). Finally, if the derivative is equal to *zero*, the tangent line is *parallel* to the X-axis (or coincides with it). Fig. 127 shows various forms of curve where this happens.

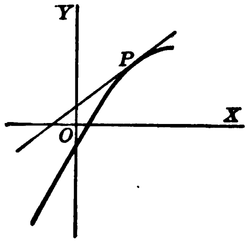


FIG. 125

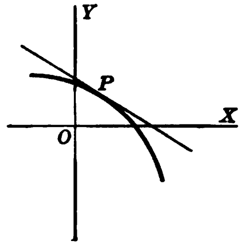


FIG. 126

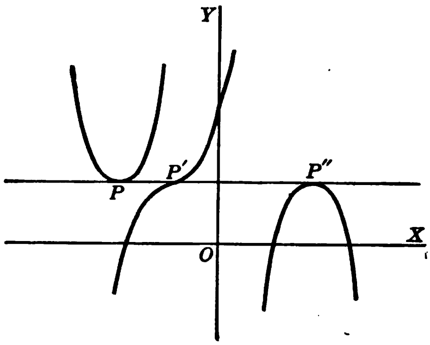


FIG. 127

Thus, in drawing the graph of a given function it will often be found useful to compute the value of the derivative, and especially to note at what points it is positive, negative, or zero.

$$\text{SUMMARY. } \textit{If } D_xy \textit{ is} \begin{cases} \textit{positive, the curve is rising.} \\ \textit{negative, the curve is falling.} \\ \textit{zero, the curve has a horizontal tangent.} \end{cases}$$

Example. Draw the graph of $y = 2x^3 + x^2 - 8$.

Making a table of corresponding values of x and y, we find

x	0	1	2	-1	-2
y	-8	-5	12	-9	-20

These points give a general idea of the shape of the curve, but we can learn more about it by making use of the derivative of y with respect to x,

$$D_x y = 6x^2 + 2x = 2x(3x + 1).$$

This expression has the value 0 for $x = 0$ and for $x = -\frac{1}{3}$, and is positive for all values of x except those in the interval from $-\frac{1}{3}$ to 0 inclusive, where it is negative.

Thus the curve is *rising* except for values of x in the interval from $x = -\frac{1}{3}$ to $x = 0$. At $x = 0$ (where $y = -8$) the tangent is horizontal, also at $x = -\frac{1}{3}$ (where $y = -7\frac{26}{27}$). Only between these points (A and B in Fig. 128) is the curve falling. This short interval would in all probability have been overlooked if we had not had the help of the derivative in locating it. At all events, we could not have been certain what were the *highest* and *lowest* points on the little wave in the curve. We now know that A is the highest point in its immediate vicinity (because the curve is *rising* on the left of A and *falling* on the right of A), and that B is the lowest point in *its* immediate vicinity (because the curve is *falling* on the left of B and *rising* on the right of B). Such a point as A, that is, a point which is the *highest point* in its immediate vicinity, is called a *maximum* point; while a point (such as B) which is the *lowest* point in its immediate vicinity is called a *minimum* point.

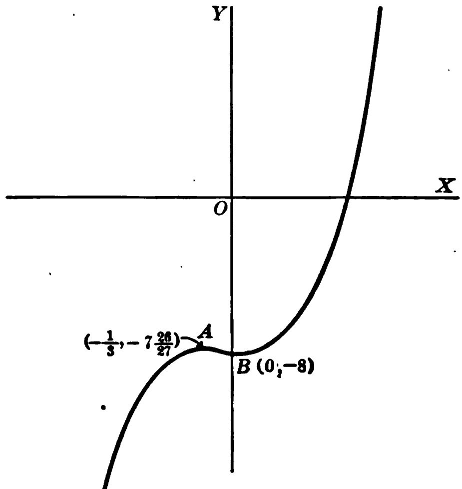


FIG. 128

176. Maximum and minimum values. Summary. Thus we see that, in general, the value $x = a$ will give a *maximum point* on the curve $y = f(x)$ when $D_x f(x)$ is zero at the point $x = a$, and is positive just to the left, and negative just to the right, of that point; that is, $f'(a - h) > 0$, $f'(a) = 0$, $f'(a + h) < 0$, h being a small positive number. For in that case $f(x)$ is increasing as x approaches a from the left, and is decreasing as x passes beyond a toward the right. Hence at $x = a$, $f(x)$ must have the greatest

value that it can have for any point in that immediate vicinity. For similar reasons the value $x = a$ will give a *minimum* on the curve $y = f(x)$ when $f'(a-h) < 0$, $f'(a) = 0$, and $f'(a+h) > 0$.

These results are summarized in the following table:

	$f'(a-h)$	$f'(a)$	$f'(a+h)$
Maximum . . .	>0	0	<0
Minimum . . .	<0	0	>0

Example. Draw the graph of $y = \dfrac{x}{1-x^2}$.

As we saw on page 87, the graph of any fractional function will have a vertical asymptote corresponding to any value of x that gives the denominator the value zero. Hence we have here the asymptotes $x = 1$ and $x = -1$. The value of the derivative will give the direction of the curve at any point:

$$D_x y = \frac{1 - x^2 - x(-2x)}{(1-x^2)^2}$$
$$= \frac{1+x^2}{(1-x^2)^2}.$$

Since this is always positive, the curve is always *rising*; hence there can be no maximum or minimum point.

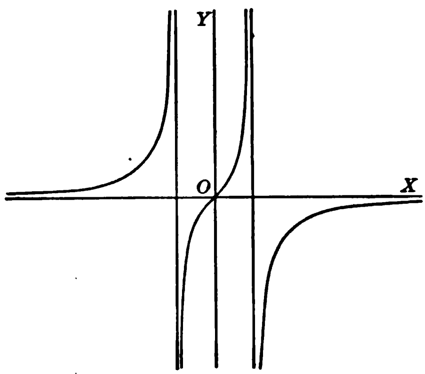


FIG. 129

EXERCISES

Draw the graphs of the following functions, making use of all the information which the derivative gives, especially with regard to maximum and minimum points:

1. $y = x^3 - 4x + 4$.
2. $y = 3x^3 - x^2 + 1$.
3. $y = x^4$.
4. $y = x^4 - x^2 + 1$.
5. $y = \dfrac{x+2}{1-x^2}$.
6. $y = \dfrac{1-x^2}{x+2}$.
7. $y = \dfrac{x^2-1}{x^2-4}$.
8. $y = \dfrac{1+x-x^2}{1+x+x^2}$.

9. Discuss the function $ax^2 + bx + c$ for maximum and minimum values. Compare the results with those obtained by more elementary methods in § 48 (p. 58).

177. Differentiation of irrational functions. The only irrational functions that we have considered have been those involving *square roots.* Any such functions can be differentiated as follows:

If $y = \sqrt{u}$, where u is any function of x that has a derivative, then

$$y + \Delta y = \sqrt{u + \Delta u}.$$

Therefore $$\Delta y = \sqrt{u + \Delta u} - \sqrt{u}$$

and $$\frac{\Delta y}{\Delta x} = \frac{\sqrt{u + \Delta u} - \sqrt{u}}{\Delta x}.$$

In this form it is not easy to see what limit the quotient $\frac{\Delta y}{\Delta x}$ will approach as $\Delta x \doteq 0$; but by multiplying both terms of the fraction by $\sqrt{u + \Delta u} + \sqrt{u}$ we get

$$\frac{\Delta y}{\Delta x} = \frac{(\sqrt{u + \Delta u} - \sqrt{u})(\sqrt{u + \Delta u} + \sqrt{u})}{\Delta x(\sqrt{u + \Delta u} + \sqrt{u})}$$

$$= \frac{(u + \Delta u) - u}{\Delta x(\sqrt{u + \Delta u} + \sqrt{u})} = \frac{1}{\sqrt{u + \Delta u} + \sqrt{u}} \cdot \frac{\Delta u}{\Delta x}.$$

Now, as $\Delta x \doteq 0$, $\frac{\Delta u}{\Delta x} \doteq D_x u$ and $\sqrt{u + \Delta u} \doteq \sqrt{u}$, since $\Delta u \doteq 0$.

Therefore $$\lim_{\Delta x \doteq 0} \frac{\Delta y}{\Delta x} = \frac{1}{2\sqrt{u}} D_x u;$$

that is, $$D_x y = \frac{1}{2\sqrt{u}} \cdot D_x u. \qquad (1)$$

In words this result is, *The derivative of the square root of a function is equal to 1 divided by twice the square root of the function, times the derivative of the function.*

Notice that (1) can be written

$$D_x(u^{\frac{1}{2}}) = \tfrac{1}{2} u^{-\frac{1}{2}} D_x u,$$

which is exactly the form we should get by using Rule IV (2) (p. 216) with $n = \frac{1}{2}$; that is, the rule $D_x(u^n) = n \cdot u^{n-1} \cdot D_x u$ holds when n has this *fractional* value, as well as for the positive integral values for which the rule was proved. As a matter of fact, the rule is still true if n has *any* fractional value, but

we do not need to use that general fact in our work. For our purposes it will usually be found simpler to use the form of equation (1) than to change to the form of a fractional exponent.

Example 1. Find the derivative of $\sqrt{2x-3}$.

Solution. $$D_x\sqrt{2x-3} = \frac{1}{2\sqrt{2x-3}}D_x(2x-3) = \frac{1}{\sqrt{2x-3}}.$$

Example 2. Find the derivative of $x\sqrt{1-x^2}$.

Solution. $$D_x(x\sqrt{1-x^2}) = \sqrt{1-x^2}\,D_x x + xD_x\sqrt{1-x^2}$$
$$= \sqrt{1-x^2} + x\cdot\frac{-2x}{2\sqrt{1-x^2}} = \frac{1-2x^2}{\sqrt{1-x^2}}.$$

EXERCISES

Find the derivatives of each of the following functions:

1. $\sqrt{5x}$.
2. $\sqrt{x^2+1}$.
3. $\sqrt{\dfrac{x}{x+1}}$.
4. $\sqrt{x^2-4}$.
5. $\sqrt{\dfrac{x+1}{x-1}}$.
6. $\sqrt{x^3}$.
7. $\sqrt{100-25x^2}$.
8. $\dfrac{x}{\sqrt{x+1}}$.
9. $\dfrac{3x}{\sqrt{4-x^2}}$.
10. $\dfrac{b}{a}\sqrt{a^2-x^2}$.
11. $x^2\sqrt{x+1}$.
12. $x+\dfrac{1}{\sqrt{x}}$.

Draw the graphs of the functions in Exs. 1–9, locating any maximum or minimum values.

178. Differentiation of implicit functions. When we have an equation in the variables x and y which determines y as an implicit function of x, it is easy to write down the value of D_xy without first solving the equation for y. Rules I–V (pp. 214–217) are all that is needed.

Example. $$x^2+4y^2=4 \tag{1}$$

(1) being true, the derivative of the left-hand side must equal the derivative of the right-hand side,
$$D_x(x^2+4y^2) = D_x(4);$$
that is,
$$D_x(x^2)+D_x(4y^2) = 0.$$
Therefore
$$2x+4\cdot 2yD_xy = 0.$$
Therefore
$$D_xy = -\frac{x}{4y}.$$

(The above work assumes that the function y has a derivative. This assumption is justified in every case with which we shall have to do.)

EXERCISES

Find D_xy from each of the following equations:

1. $x^2 + y^2 = 9$. *Ans.* $D_xy = -\dfrac{x}{y}$.
2. $4x^2 + y^2 = 4$.
3. $9x^2 - 4y^2 = 36$.
4. $xy + y^2 = 4$. *Ans.* $D_xy = -\dfrac{y}{x+2y}$.
5. $x^2 - 2xy + y^2 + 4x - 8y = 0$.
6. $x^3 + y^3 = 65$.
7. $x^2y + xy^2 = 25$.
8. $(x-\alpha)^2 + (y-\beta)^2 = r^2$.
9. $b^2x^2 + a^2y^2 = a^2b^2$.
10. $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$.
11. $\dfrac{(x-\alpha)^2}{a^2} + \dfrac{(y-\beta)^2}{b^2} = 1$.
12. $ax^2 + bxy + cy^2 + dx + ey + f = 0$.

13. Prove that the equation of the tangent to the parabola $y^2 = 4ax$ at the point (x_1, y_1) is $yy_1 = 2a(x + x_1)$.

14. Obtain the equation of the tangent to the hyperbola $xy = 1$ at the point (x_1, y_1) in the form $y_1x + x_1y = 2$.

15. Show that the hyperbola of Ex. 14 and the circle $x^2 + y^2 = 2$ have the same tangent at their common points; that is, that they are tangent to each other.

16. At what point on the ellipse $4x^2 + 9y^2 = 36$ is the tangent parallel to the line $y = -x$?

17. Verify the results of Ex. 17, § 109, and Ex. 2, § 110, by finding the value of D_xy in each case.

18. Prove that the tangents to a parabola at the ends of the latus rectum intersect at right angles on the directrix.

179. Summary. We have now seen how to find the derivative of any rational function, and of irrational functions of degree not higher than the second. Since the derivative of y with respect to x gives the slope of the tangent at a point, we can find the equation of the tangent to any curve that is the graph of one of these functions.

The derivative gives also the rate of change of the function per unit change of the independent variable, so that this important problem is solved for all the algebraic functions mentioned. We have accordingly reached the point where we can consider the applications of this work to problems from various fields.

Example 1. The Law of Falling Bodies.

If
$$s = 16\,t^2 \tag{1}$$
states the relation between distance and time in the case of a falling body, then the instantaneous rate of change of the distance with respect to the time, that is, the *velocity* of falling, is given by $D_t s$. Since $D_t s = 32\,t$, we have at once the formula
$$v = D_t s = 32\,t, \tag{2}$$
which is well known from elementary physics, where it appears as a second formula, as if it were independent of (1). But we now see that it is not, since, if (1) is true, (2) must be true also.

Example 2. If a body moves according to the law $s = t^2 - 10\,t + 2$, find its velocity at any instant t. When is it moving in the positive direction, and when in the negative?

Solution. The velocity is given by $D_t s = 2\,t - 10$, which is negative until $t = 5$, when it equals zero; for $t > 5$ it is positive. Hence the body is moving in the negative direction until $t = 5$, when it has zero velocity, after which it moves in the positive direction.

Example 3. A ladder 40 ft. long rests against the side of a house. If its foot A is pulled away from the house at the rate of 5 ft. per second, how fast is the top of the ladder descending when the bottom is 10 ft. from the house?

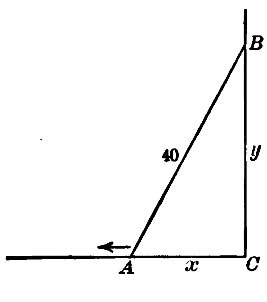


FIG. 130

Solution. Let $x = AC$, the distance from the house to the foot of the ladder; and let $y = CB$, the height of the top of the ladder. Then y is a function of x, and the problem is to find the rate of change of y, knowing that of x. From the right triangle ABC
$$y^2 = 1600 - x^2.$$

Therefore
$$2\,yD_x y = -\,2\,x,$$
or
$$D_x y = -\frac{x}{y};$$
that is, y is changing at any instant $-\dfrac{x}{y}$ times as fast as x is. We are to find the rate of change of y when $x = 10$. When $x = 10$,
$$y = \sqrt{1600 - 100} = \sqrt{1500} = 10\sqrt{15};$$
hence
$$D_x y = -\frac{10}{10\sqrt{15}} = -\frac{1}{\sqrt{15}}.$$

Therefore y is at that instant changing $-\frac{1}{\sqrt{15}}$ times as fast as x. But, by the conditions of the problem, x is changing at the rate of 5 ft. per second, and accordingly the rate of change of y is $-\frac{1}{\sqrt{15}}\cdot 5 = -\frac{1}{3}\sqrt{15}$ ft. per second. The negative sign shows that y is *diminishing*.

EXERCISES

1. Two persons start from the same point and walk in directions at right angles to each other at the rates of 3 mi. per hour and 4 mi. per hour respectively. How fast are they separating after 15 min. has elapsed?

2. A light is 10 ft. above a street crossing, and a man 6 ft. tall walks away from it in a straight line at the rate of 4 mi. per hour. How fast is the tip of his shadow moving after 10 sec.? after 1 min.? How fast is the shadow lengthening in each case?

3. A point moves along the parabola $y^2 = 8x$. How does the rate of change of the ordinate at the point $(\frac{1}{2}, 2)$ compare with that of the abscissa?

4. At what point on the parabola of Ex. 3 are the ordinate and the abscissa changing at the same rate?

5. When a stone is thrown into still water, how fast does the area of the circle formed by the ripples change in comparison with the radius?

6. If the radius of a spherical soap bubble is 3 in. and is increasing at the rate of $\frac{1}{2}$ in. per second, at what rate is the volume increasing?

7. If a point moves along the curve $x^2 + 2y^2 = 2$, how does the rate of change of the ordinate compare with that of the abscissa at the point $(\frac{4}{3}, \frac{1}{3})$?

8. An automobile track is in the form of an ellipse whose major axis is $\frac{1}{2}$ mi. long and whose minor axis is $\frac{1}{4}$ mi. long. If a car moves around the track at the rate of 40 mi. per hour, how fast is it moving toward the north, and how fast toward the east when it is at the end of the latus rectum? (Suppose the length of the track to be from north to south.)

9. A man on a wharf is pulling in his boat by means of a rope fastened to its prow. If his hand is 10 ft. above the boat and he pulls in the rope at the rate of 3 ft. per second, how fast is the boat coming in when 20 ft. away? when 5 ft. away?

10. A railroad crosses a road at right angles by an overhead crossing 30 ft. high. If a train and an auto cross at the same time, the one at the rate of 30 mi. per hour, the other at the rate of 15 mi. per hour, how fast will they be separating after 1 min.?

11. One ship sails east at the rate of 10 mi. per hour, another north at the rate of 12 mi. per hour. If the second crosses the track of the first at noon, the first having passed the same point 2 hr. before, how is the distance between the ships changing at 1 P.M.? How was it at 11 A.M.? When was the distance between them not changing?

180. Problems involving maxima and minima. As we have seen, the use of the derivative of a function enables us to discover for what values of the independent variable the function will have a maximum or a minimum value. This fact is of use in solving a great variety of problems.

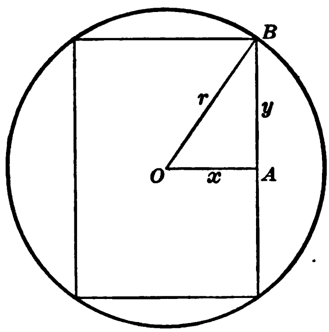


FIG. 131

Example 1. Find the rectangle of greatest area which can be inscribed in a given circle.

Solution. Let r be the radius of the given circle, and $2x$ and $2y$ the dimensions of the rectangle (Fig. 131). Then $y^2 = r^2 - x^2$ or $y = \sqrt{r^2 - x^2}$. The area of the rectangle is $4xy$, which is to be made the greatest possible by choosing the proper value of x (and hence of y, which is a function of x). Now $4xy = 4x\sqrt{r^2 - x^2} = f(x)$, so that the question is, For what value of x will $f(x)$ have its maximum value? For that value of x, as we have seen (p. 220), we may expect to find $D_x f(x) = 0$; that is,

$$D_x[4x\sqrt{r^2 - x^2}] = 0.$$

Therefore $$D_x[x\sqrt{r^2 - x^2}] = 0.$$

But $$D_x[x\sqrt{r^2 - x^2}] = \sqrt{r^2 - x^2} - \frac{x^2}{\sqrt{r^2 - x^2}} = \frac{r^2 - 2x^2}{\sqrt{r^2 - x^2}} = 0.$$

Therefore $$r^2 - 2x^2 = 0,$$

$$x = \frac{r}{\sqrt{2}}.$$

When $x < \frac{r}{\sqrt{2}}$, $D_x f(x) > 0$, and the function is *increasing* as x increases; when $x > \frac{r}{\sqrt{2}}$, $D_x f(x) < 0$, and the function is *decreasing* as x decreases.

That is, the function is *increasing* before x reaches the value $\frac{r}{\sqrt{2}}$, and *decreasing* after x passes the value $\frac{r}{\sqrt{2}}$. Hence that value of x gives a *maximum* value of $f(x)$; that is, of the area of the rectangle. Geometrically it is also evident that this value is a *maximum* rather than a *minimum*, because as x increases from very small values the area of the rectangle also increases, but beyond a certain point it will diminish again, approaching 0 as x approaches r. The locating of this "certain point" as being the value $x = \frac{r}{\sqrt{2}}$ is made possible by the use of the derivative.

Notice that when $x = \frac{r}{\sqrt{2}}$, $y = \frac{r}{\sqrt{2}}$, so that the maximum rectangle is a square.

Example 2. What would be the dimensions of a cylindrical steel tank, open at the top, to require the least material possible and still contain 50 gallons?

Solution. The quantity of material used will be least when the area of the surface is least. The surface consists of the side, whose area is $2\pi xy$, and the bottom, whose area is πx^2; total, $S = 2\pi xy + \pi x^2$ (x representing the radius of the base and y the altitude, as in Fig. 132). In this expression y is a function of x, whose value is determined by the fact that the volume is to be 50 gal.; that is, 6.684 cu. ft. The volume of the cylinder $= \pi x^2 y$.

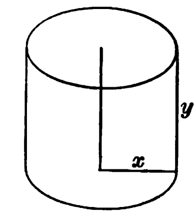


FIG. 132

Therefore $$\pi x^2 y = 6.684,$$

$$y = \frac{6.684}{\pi x^2}.$$

Therefore $$S = 2\pi x \cdot \frac{6.684}{\pi x^2} + \pi x^2 = \frac{13.368}{x} + \pi x^2.$$

This function of x is to be made a minimum.

Therefore $$D_x S = 0.$$

$$D_x S = -\frac{13.368}{x^2} + 2\pi x = 0.$$

$$x^3 = \frac{13.368}{2\pi},$$

$$x = \sqrt[3]{\frac{6.684}{\pi}} = 1.286, \text{ approximately.}$$

From the nature of the problem this value makes the function S a minimum, not a maximum (as can also be verified by noting that $D_x S$

changes sign from $-$ to $+$ as x increases through the value 1.286), so that the value $x = 1.286$ gives the radius of the base that will require the least amount of material for the tank.

When $$x = 1.286,$$

$$y = \frac{6.684}{\pi x^2} = 1.286, \text{ approximately};$$

that is, $$y = x.$$

The exact value of x is $\sqrt[3]{\frac{6.684}{\pi}}$, which equals $\sqrt[3]{x^2 y}$, so that $x = \sqrt[3]{x^2 y}$. Therefore $x = y$ exactly, in fact.

Thus the height of the tank should equal the diameter of the base in order to require the least amount of material.

EXERCISES

1. Work through Ex. 2 above, using V as the volume of the tank, instead of 50 gal. Show that $y = x$ will give the proportions to require the least material.

2. Divide a length AB into two parts so that the product of the lengths of the segments shall be the greatest possible (Fig. 133.)

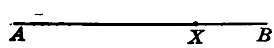


FIG. 133

3. Divide a length AB into two parts so that the sum of the squares of the lengths of the segments shall be the least possible.

4. A piece of cardboard 10 in. square has a small square cut out at each corner, and the remainder is folded up so as to form a box. How large a square should be cut out so that the box may be the largest possible? *Ans.* $\frac{5}{3}$ in. is the side of the square cut out.

5. A rectangular piece of cardboard, 15 in. $\times$ 7 in., is to be made into a box by cutting out a square of equal size from each corner. How large should this square be in order that the box may have the largest possible contents?

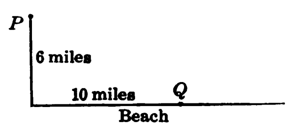


FIG. 134

6. A person in a boat, 6 miles from the nearest point of the beach, wishes to reach in the shortest possible time a place 10 miles from that point along the shore; if he can walk 5 mi. per hour, and row 4 mi. per hour, where should he land? (Fig. 134.)

7. A rectangular box, open at the top, is to be constructed to contain a certain volume V. What must be its dimensions in order to require the least amount of material?

8. Find the largest rectangle that can be inscribed in an isosceles triangle whose base is 4 in. and whose slant height is 10 in. (Fig. 135).

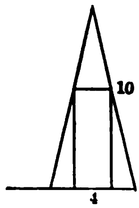


FIG. 135

9. Find the largest cylinder that can be inscribed in a given sphere.

10. Find the largest cone that can be inscribed in a given sphere.

11. What should be the dimensions of a cylindrical tin can (both ends being closed) in order to require the least material?

12. The lower corner of a sheet of paper whose width is a is folded over so as just to reach the other edge of the paper. How wide should the part folded over be in order that the length of the crease may be the minimum?

13. Find the shortest and the longest distance from the point (4, 6) to the circle $x^2 + y^2 = 25$.

14. Find the shortest distance from the point (0, 1) to the hyperbola $xy = 1$. Show that this distance is measured along the *normal* to the curve through the given point.

15. If the cost of running a steamboat is proportional to the cube of the velocity generated, what is the most economical rate of steaming against a 3 mi. per hour current?

PORTION OF GREEK ALPHABET

Alpha	α	Lambda	λ
Beta	β	Pi	π
Gamma	γ	Rho	ρ
Delta	Δ δ	Tau	τ
Epsilon	ϵ	Phi	ϕ
Theta	θ	Omega	ω

APPENDIX A

PROOF THAT THE DIAGONAL OF A SQUARE IS INCOMMENSURABLE WITH ITS SIDE

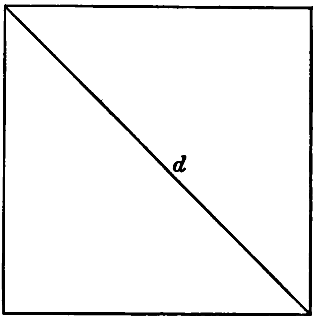
This theorem proves the fact asserted in the text (p. 3), that there exist segments which cannot be measured in terms of a specified unit. The fact to be proved may be stated in the following words:

If the side of a square be taken as unit, there exists no number $\frac{m}{n}$ (m and n being integers) such that the length of the diagonal of the square is $\frac{m}{n}$.

Proof. By the Pythagorean Theorem,

$$d^2 = 1^2 + 1^2 = 2. \tag{1}$$

We can prove that the assumption $d = \frac{m}{n}$, where m and n are integers, leads to a contradiction. Suppose $\frac{m}{n}$ to be in its lowest terms; that is, suppose m and n to have no common divisor.



Then, since $d^2 = 2$,

$$\frac{m^2}{n^2} = 2.$$

Therefore $$m^2 = 2\,n^2. \tag{2}$$

Therefore m^2 is divisible by 2. This is only possible if m is an *even* integer (since the square of an *odd* integer is *odd*); and since m is even and $\frac{m}{n}$ is in its lowest terms, n must be *odd*.

Further, since m is even, we can write $m = 2\,k$, where k is some integer.

Therefore $$m^2 = 4\,k^2. \tag{3}$$

Comparing (3) with (2), $$2\,n^2 = 4\,k^2;$$

that is, $$n^2 = 2\,k^2. \tag{4}$$

Hence n^2, and therefore n, is *even*.

But we proved above that n is *odd*; hence a *contradiction* results from the assumption that $d = \frac{m}{n}$. Therefore the assumption is false, and there does *not* exist any number $\frac{m}{n}$ representing the diagonal of a square whose side is the unit.

NOTE. This proof is of ancient origin, being given in almost exactly the above form by Euclid, in his famous "Elements of Geometry," in the third century B.C., and being referred to by Aristotle, as well known, much earlier. It is very probable that it was discovered by Pythagoras himself in connection with the study of his famous theorem (see p. 4, footnote).

APPENDIX B

LAWS OF OPERATION WITH RADICALS

The following are the laws of operation with irrational numbers that are in the form of radicals:

I. **Multiplication.** $\sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{ab}.$

Special case. $\sqrt{a} \cdot \sqrt{b} = \sqrt{ab}.$

Examples. $\sqrt{2} \cdot \sqrt{3} = \sqrt{6}$; $\sqrt{12} = \sqrt{4} \cdot \sqrt{3} = 2\sqrt{3}.$

II. **Division.** $\dfrac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\dfrac{a}{b}}$; but in no case may the denominator equal 0, since division by 0 is an impossible operation.

Examples. $\sqrt{\dfrac{3}{4}} = \dfrac{\sqrt{3}}{2}$; $\sqrt{\dfrac{2}{3}} = \sqrt{\dfrac{6}{9}} = \dfrac{1}{3}\sqrt{6}$; $\sqrt{\dfrac{1}{5}} = \sqrt{\dfrac{5}{25}} = \dfrac{1}{5}\sqrt{5}.$

III. **Addition and subtraction.** Only *similar* radicals can be added or subtracted; that is, radicals that involve the *same root of the same number.*

Examples. $\sqrt{2} + \sqrt{3}$ cannot be simplified, nor can $\sqrt{2} + \sqrt[3]{2}$, but $\sqrt{2} + \sqrt{18} = \sqrt{2} + 3\sqrt{2} = 4\sqrt{2} = \sqrt{32}.$

(It is a very common error to write $\sqrt{a+b} = \sqrt{a} + \sqrt{b}$, which violates this law.)

EXERCISES

Give the value of each of the following in the simplest form:

1. $\sqrt{\frac{1}{5}}$.
2. $\dfrac{2-\sqrt{3}}{2+\sqrt{3}}$.
3. $\dfrac{2-3\sqrt{2}}{3+2\sqrt{2}}$.
4. $(2-\sqrt{3})(2+\sqrt{3})$.
5. $\dfrac{\sqrt{x+3}-\sqrt{x-3}}{\sqrt{x+3}+\sqrt{x-3}}$.
6. $\dfrac{xy}{1-\sqrt{1-x^2y^2}}$.
7. $\dfrac{\sqrt{6}+\sqrt{2}}{2+\sqrt{3}}$.
8. $\dfrac{4-\sqrt{6}-\sqrt{2}}{\sqrt{6}-\sqrt{2}}$.

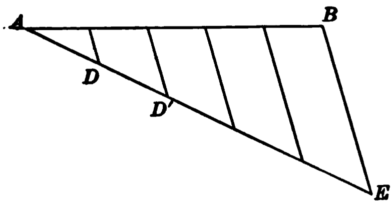
HINT. A fraction containing $\sqrt{a}+\sqrt{b}$ in the denominator is simplified by multiplying both numerator and denominator by $\sqrt{a}-\sqrt{b}$. Accordingly, in Ex. 3, we should multiply both terms of the fraction by $2-\sqrt{3}$.

9. Prove: (a) $\dfrac{1}{\sqrt{2}} = \dfrac{\sqrt{2}}{2}$. (b) $\dfrac{1}{\sqrt{2}-1} = 1+\sqrt{2}$. (c) $\dfrac{5\sqrt{5}-10}{5\sqrt{5}+10} = 9-4\sqrt{5}$. (d) $\sqrt{3-2\sqrt{2}} = (\sqrt{2}-1)$.

APPENDIX C

TO CONSTRUCT A SEGMENT HAVING A RATIONAL LENGTH

1. *To divide a given segment AB into any number of equal parts:* Draw any line AE making an angle with AB. On AE lay off any length, as AD, n times, and let the last division point be E. Draw



EB, and from all the other division points draw lines parallel to EB. These parallels intersect AB in the points of division required, and one of the resulting segments is $\frac{1}{n}$ of AB. (Why?)

EXERCISES

Take a random segment as unit, and construct $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$, $\frac{1}{6}$, $\frac{4}{3}$, $1\frac{5}{6}$, $2\frac{3}{4}$, 0.6, 3.1, 5.3. Take two random segments, a and b. Construct $\frac{1}{2}a + \frac{1}{3}b$, $3a + 4b$, $1\frac{2}{3}a + 2\frac{1}{2}b$.

2. *To divide a given segment AB into two segments having any given ratio, as $m : n$.*

This problem is solved by a method very similar to that above, and it is accordingly left to the student as an exercise.

APPENDIX D

TO CONSTRUCT THE SQUARE ROOT OF ANY GIVEN SEGMENT

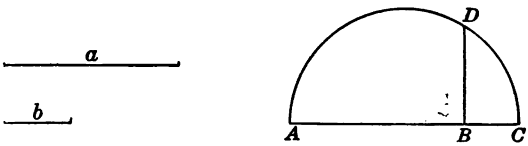
This important problem of construction is solved by means of the following

THEOREM. *In a right triangle the altitude drawn from the vertex of the right angle is the mean proportional between the segments into which it divides the hypotenuse.*

The proof is found in any textbook of elementary geometry.

By using this theorem we can construct the mean proportional between any two given segments, as follows:

Let a and b be the two segments. Draw $AB = a$, and produce it to C so that $BC = b$. Draw a semi-circle on AC as diameter. At B



erect the perpendicular to AC, meeting the semi-circle in D. Then BD is the mean proportional between AB and BC; that is, between the given segments a and b.

The proof follows at once from the fact that the angle ADC is a right angle.

Now, to construct the square root of any given segment m, construct as above the mean proportional to m and 1. This mean proportional then has the length $\sqrt{m \cdot 1}$; that is, it is the required $\sqrt{m}$.

EXERCISE

Take any random segment and construct its square root.

APPENDIX E

SIMULTANEOUS LINEAR EQUATIONS IN TWO UNKNOWN QUANTITIES

If we have the two equations

$$\begin{cases} 3x - 7y = 1, & (1) \\ 2x + y = 12, & (2) \end{cases}$$

which are both to be satisfied by a certain pair of values (x, y), we proceed to combine the two equations in such a way as to obtain a new equation which shall contain only one unknown quantity. To do this we may add (1) and (2) as they stand or after they have been multiplied by any number we please. If we multiply (2) by 7, and add the result to (1), it is clear that we shall get rid of the y-term altogether; we call this "eliminating y" from (1) and (2).

Thus, (2) multiplied by 7 gives

$$14x + 7y = 84.$$

Adding this to (1), $\qquad 17x = 85.$

Therefore $\qquad x = 5.$

If we substitute this value of x in (1), we get

$$15 - 7y = 1.$$

$$7y = 14.$$

$$y = 2.$$

Therefore the pair of values (5, 2) will satisfy both equations.

We could also have eliminated x from (1) and (2) by multiplying (1) by 2 and (2) by 3 and subtracting:

$$6x - 14y = 2,$$

$$6x + 3y = 36.$$

Therefore $\qquad 17y = 34,$

$$y = 2.$$

Substituting $y = 2$ in (1),

$$3x - 14 = 1.$$
$$3x = 15.$$
$$x = 5.$$

Therefore (5, 2) is the solution. This result should be checked by substituting $x = 5$, $y = 2$ in (1) and (2).

It is clear that this procedure can always be adopted, giving a new equation in which one or the other of the two unknowns fails to appear.

APPENDIX F

THE QUADRATIC EQUATION IN ONE UNKNOWN QUANTITY

Suppose we wish to solve the equation $x^2 + 6x = 7$. The method of solution most commonly employed in elementary algebra is that known as "completing the square," because it consists in adding to the left side of the equation such a number that that side becomes a perfect square. Success in using this method therefore depends upon familiarity with the form that an expression must have if it is a perfect square. We know that $(x + k)^2 = x^2 + 2kx + k^2$, which shows that when the terms of a complete square are written in this order, the middle term $(2kx)$ is *twice the product of the square roots* of the other two terms. As one of these square roots is x (the first term being x^2), the other one must be *half the coefficient of* x in the middle term. Thus, in the equation $x^2 + 6x = 7$ above, $x^2 + 6x + k^2$ will be a perfect square if $6 = 2k$, that is, if $k = 3$; hence the quantity to be added to $x^2 + 6x$ to make it a perfect square is 3^2; $x^2 + 6x + 9 = (x + 3)^2$.

Similarly, to $x^2 + 8x$ we must add 4^2 to complete the square, to $x^2 + 5x$ we must add $\left(\frac{5}{2}\right)^2$, and to $x^2 + mx$ we must add $\left(\frac{m}{2}\right)^2$. In words, *The quantity to be added is the square of half the coefficient of* x, *when the coefficient of* x^2 *is 1.*

EXERCISES

Complete the square of each of the following:

$x^2 + 10x$. $\quad x^2 + (m + n)x$. $\quad x^2 + \frac{4}{7}x$.

$x^2 + 15x$. $\quad x^2 + ax$.

$x^2 + \frac{3}{4}x$. $\quad x^2 + \frac{1}{2}x$. $\quad x^2 + \frac{2ab}{a + b}x$.

Solve each of the following equations:

1. $x^2 + 6x = 7$.

Solution. To complete the square of the left side we must add 9:

$$x^2 + 6x + 9 = 16.$$

Extracting the square root, $\quad x + 3 = \pm 4$;

that is, $\quad x = 1 \text{ or } -7$.

2. $x^2 + 5x = 6$.
3. $x^2 - 5x = 6$.
4. $x^2 + 7x = -12$.
5. $x^2 + 12x = -20$.
6. $x^2 + 12x = \frac{25}{4}$.
7. $x^2 - (a+b)x = -ab$.
8. $x^2 - 2mx + m^2 - n^2 = 0$.
9. $abx^2 = (a^2 - b^2)x + ab$.
10. $x^2 - (a+b)x = 2a^2 + 2b^2 - 5ab$.
11. $x^2 - (a+c)x + \dfrac{a^2 - b^2 + 2ac + c^2}{4} = 0$.

INDEX